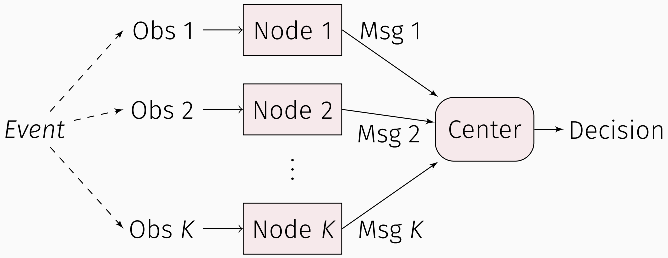
On Distributed Hypothesis Testing with Constant Communication Bits

Xiangxiang Xu

April 15, 2022

Joint work with Shao-Lun Huang

Background: Distributed Inference



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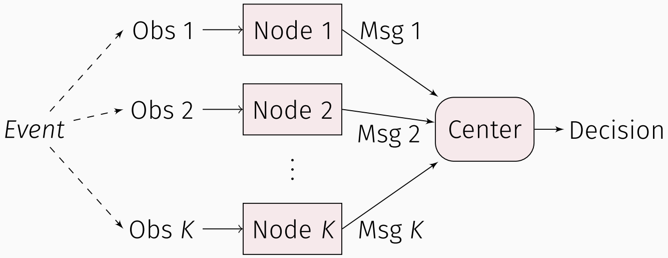

- Distributed Detection

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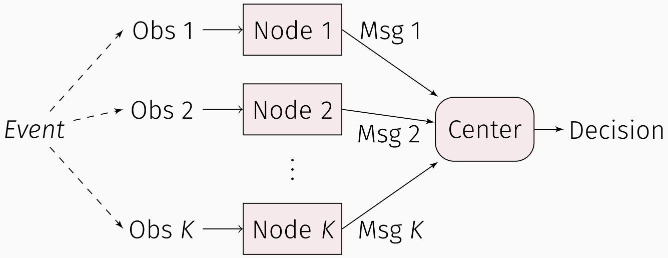

- Distributed Detection
- Sensor Fusion

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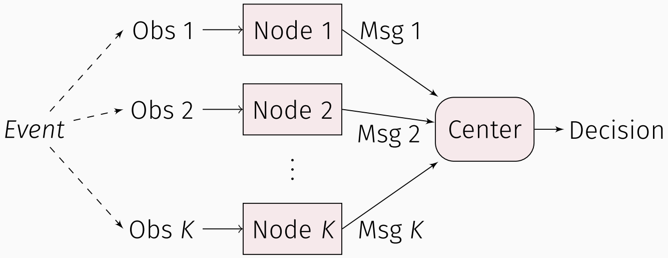
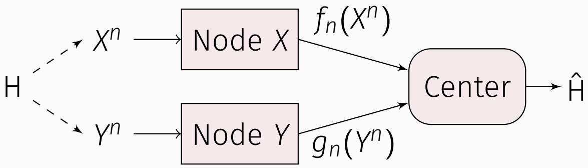
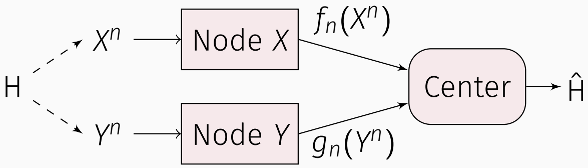


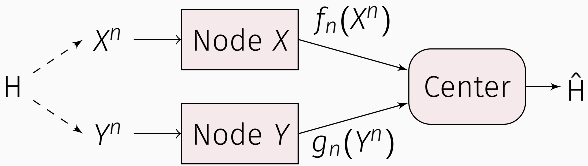
- Distributed Detection
- Sensor Fusion
- Federated Learning


H
X^n
Y^n
Node X
Node Y
$f_n(X^n)$
$g_n(Y^n)$
Center
$\hat{H}$

Problem Formulation: Distributed Hypothesis Testing (DHT)



- $H \in \{0, 1\}$: Binary Hypothesis



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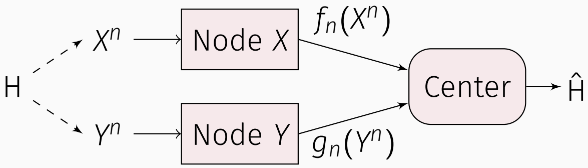

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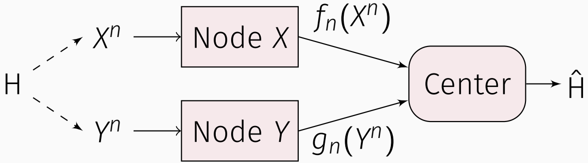

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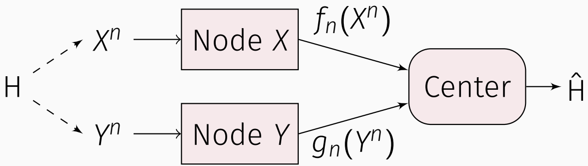

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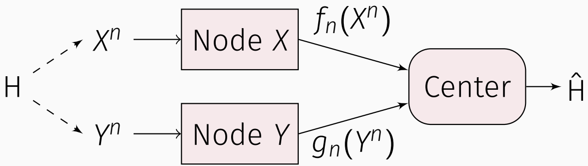

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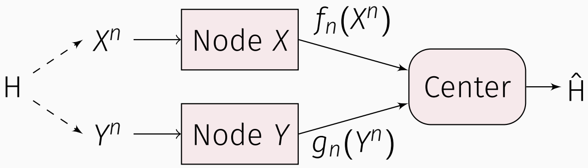
- f_n, g_n: Distributed Encoders
- $\mathrm{dec}(\cdot,\cdot)$: Central Decoder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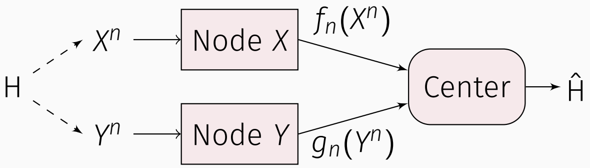

Type-I Error

$$\pi_0 \triangleq \mathbb{P}\left\{\hat{H} \neq 0 \middle| H = 0\right\}$$

Type-II Error

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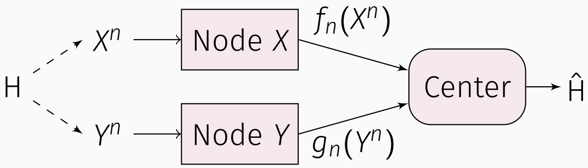

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Error exponent E_0: $\pi_0 \doteq \exp(-nE_0)$

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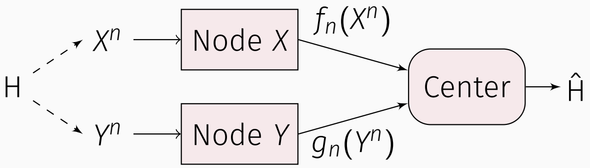
Performance Metrics: Errors and Error Exponents



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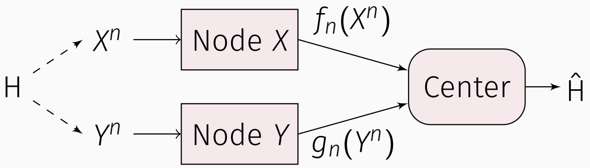
Optimal E_1 under $\pi_0 < \epsilon$



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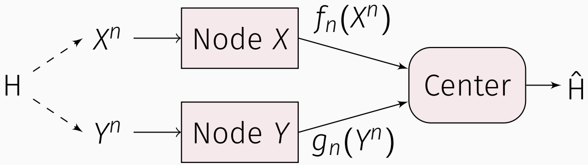
Optimal E_1 under $\pi_0 < \epsilon$ or **Optimal (E_0, E_1) Trade-offs**

Communication Constraints in DHT



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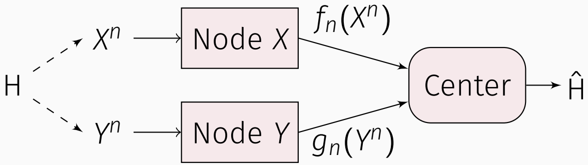
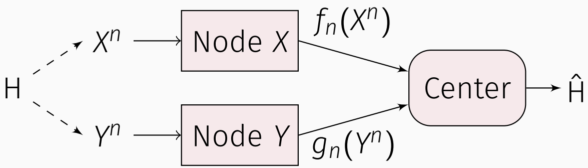


$\|f_n\|$, $\|g_n\|$: sizes of message sets



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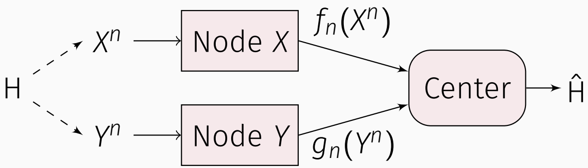
Rate Constraints (R_X, R_Y)

$$\|f_n\| \dot{\le} \exp(nR_X), \|g_n\| \dot{\le} \exp(nR_Y)$$



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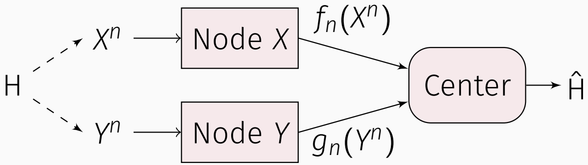
e.g. $(0, 0)$ *zero-rate compression*



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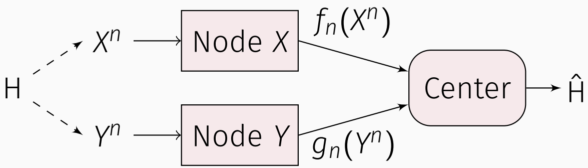
Constant Constraints $(0_{M_X}, 0_{M_Y})$

$$\|f_n\| \leq M_X, \|g_n\| \leq M_Y$$



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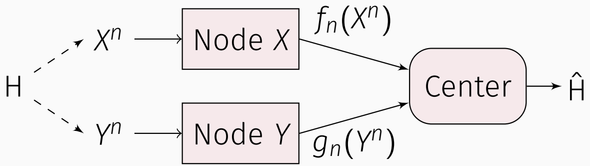
Constant Constraints $(0_{M_X}, 0_{M_Y})$

$$\|f_n\| \leq M_X, \|g_n\| \leq M_Y$$

e.g. $(0_2, 0_2)$ *one-bit compression*

Existing Results



Constraints	Opt. E_1 under $\pi_0 < \epsilon$	Opt. (E_0, E_1) Trade-offs
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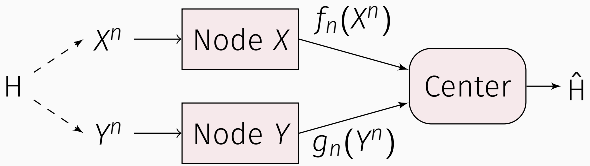

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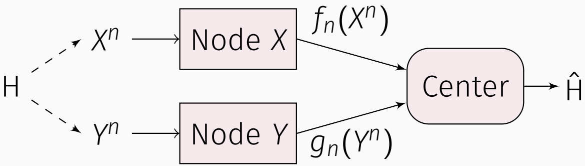

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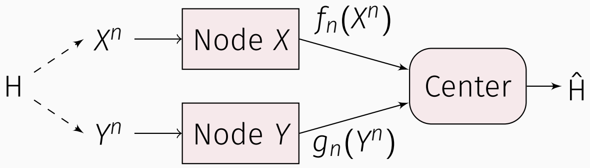

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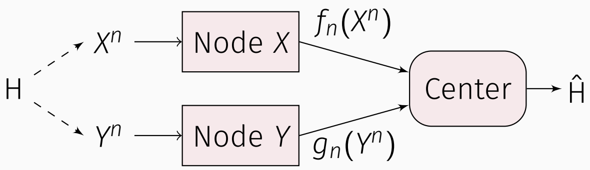

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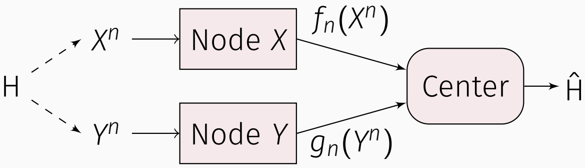

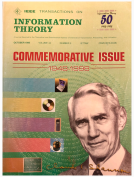

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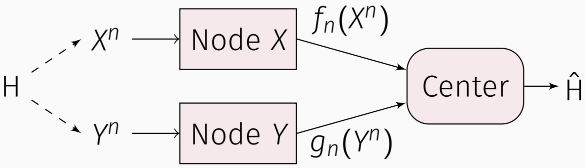

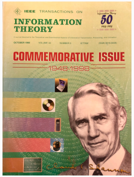

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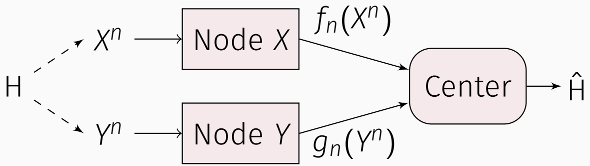

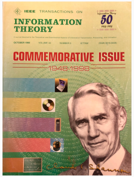

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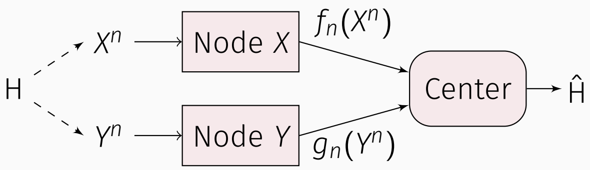

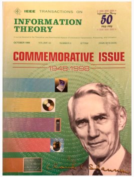

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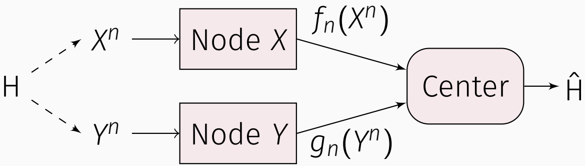

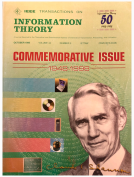

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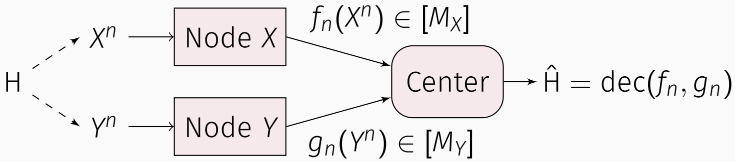
$(0_{M_X}, 0_{M_Y})$ — Constant Communication Bits



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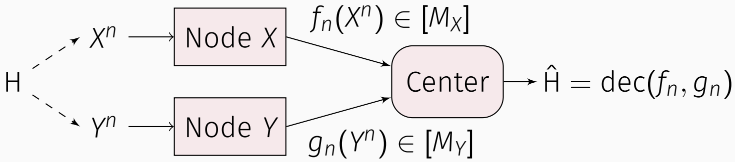
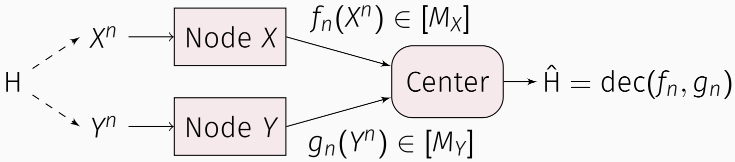


$[M] \triangleq \{0, 1, \ldots, M-1\}$: Message Set with M Symbols

$(0_{M_X}, 0_{M_Y})$ — Constant Communication Bits



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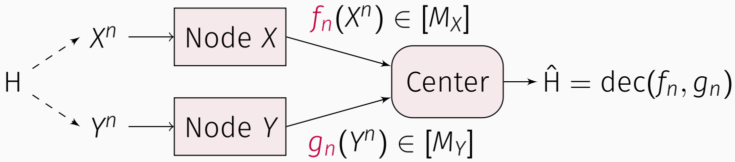

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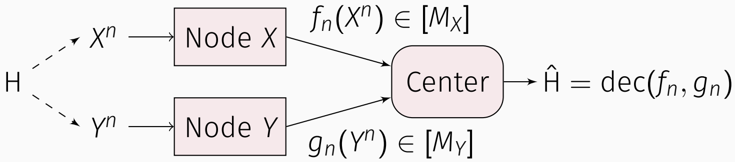

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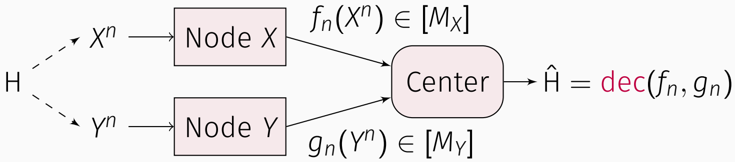

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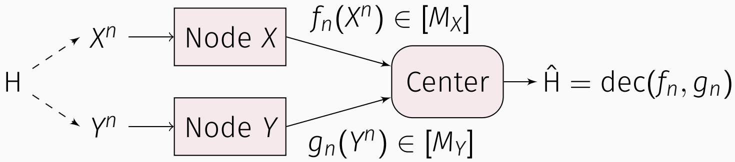

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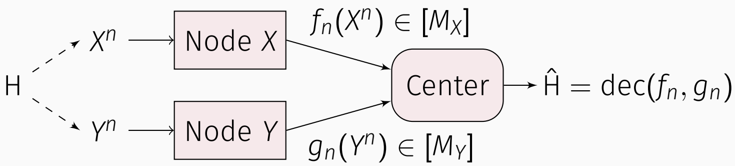

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Distributed Processing & **Communication Constraints**

Sequence $=$ Type $+$ "Order"

Encoder Design: Information Extraction from Sequences

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Example: binary sequences of length $n = 3$

Type $\leftrightarrow$ # of ones	0	1	2	3
Sequences	○○○	●○○ ○●○ ○○●	○●● ●○● ●●○	●●●

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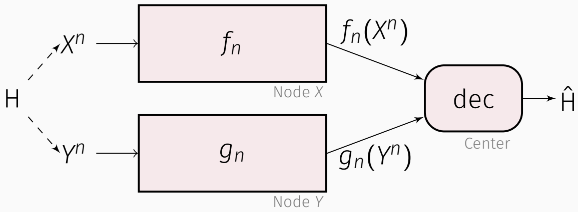
Type Requires Much Less Bits to Describe

Type-based Encoders



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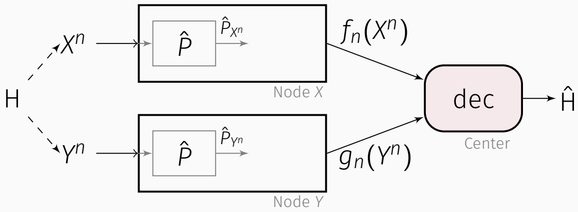

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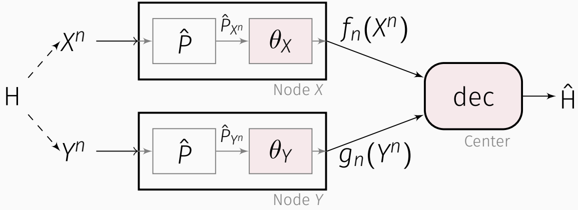

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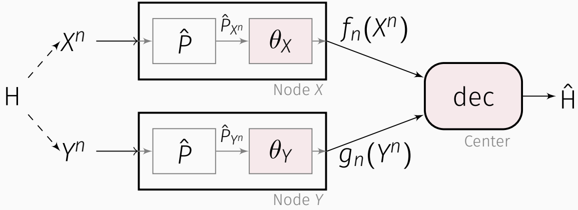

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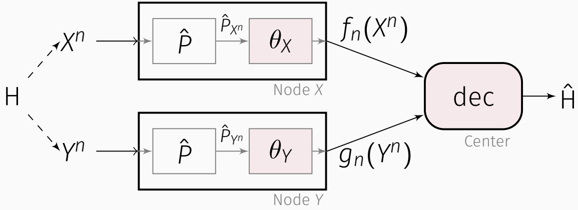

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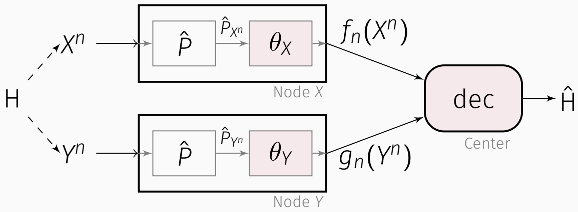

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Optimality?

Optimality of Type-based Encoders (for all decoders!)

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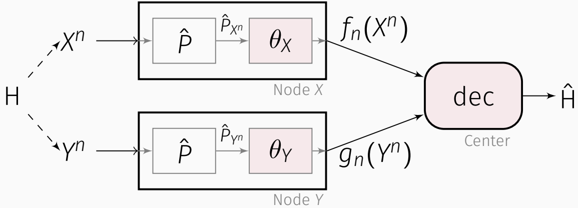
Proof An application of the blowing up lemma.

Coding Scheme Design

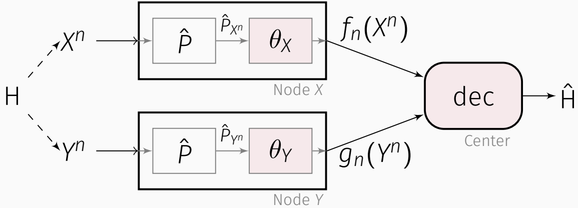


- Encoding

$$\mathcal{P}^{\mathcal{X}} \longrightarrow [M_X], \mathcal{P}^{\mathcal{Y}} \longrightarrow [M_Y]$$

- Decoding

$$[M_X] \times [M_Y] \longrightarrow \{0, 1\}$$



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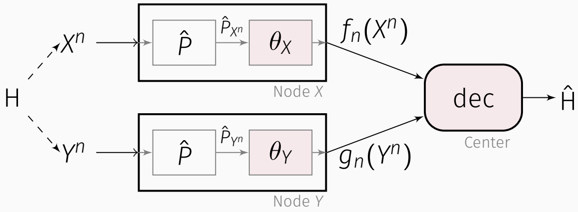

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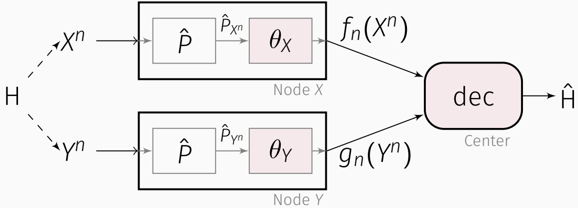

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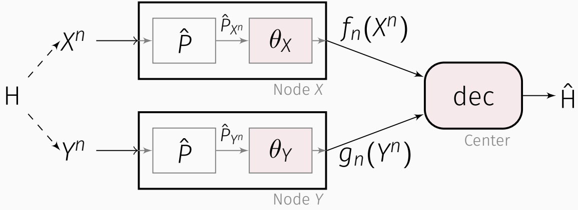

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How to design encoders (quantize types)?

One-bit Quantization of Types

Hypothesis Testing on $\mathcal{X} = \{0, 1, 2\}, P_X^{(0)} = (1/8, 1/8, 3/4), P_X^{(1)} = (3/8, 3/8, 1/4)$

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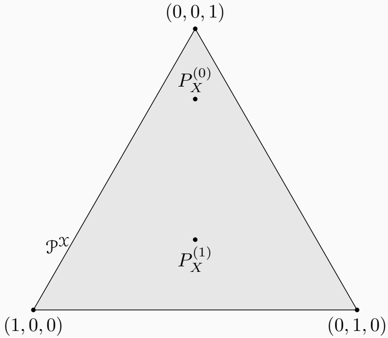

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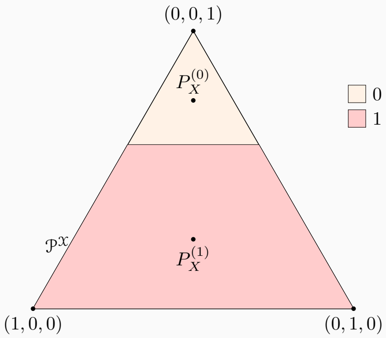
Neyman-Pearson Test



Quantize LLR: $\mathbb{E}_{\hat{P}_{X^n}} \left[\log \frac{P_X^{(0)}}{P_X^{(1)}} \right] \underset{\hat{\mathsf{H}}=1}{\overset{\hat{\mathsf{H}}=0}{\gtrless}} T$

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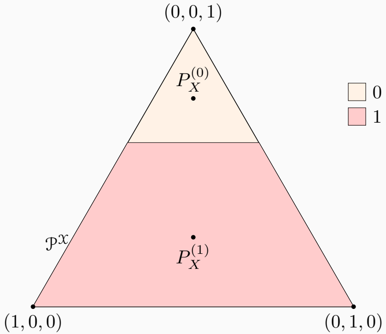

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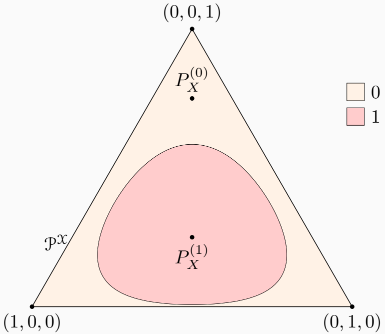
Hoeffding's Test



Quantize KL: $D(\hat{P}_{X^n} \| P_X^{(1)}) \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} T$

$$D_i^*(Q_X, Q_Y) \triangleq \min_{\substack{Q_{XY}:\ [Q_{XY}]_X = Q_X \\ [Q_{XY}]_Y = Q_Y}} D(Q_{XY} \| P_{XY}^{(i)})$$

Sequences with marginal types (Q_X, Q_Y)

- Associated probability:

$$\mathbb{P}\left\{(\hat{P}_{X^n}, \hat{P}_{Y^n}) = (Q_X, Q_Y) \middle| \mathrm{H} = i\right\} \doteq \exp\left(-n \cdot D_i^*(Q_X, Q_Y)\right)$$

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(E_0, E_1) **is achievable, iff** $\hat{\mathsf{H}}(\mathcal{D}_0(E_0)) = 0, \hat{\mathsf{H}}(\mathcal{D}_1(E_1)) = 1$

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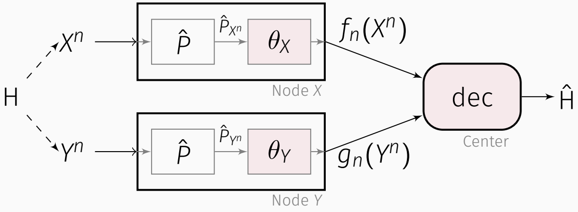

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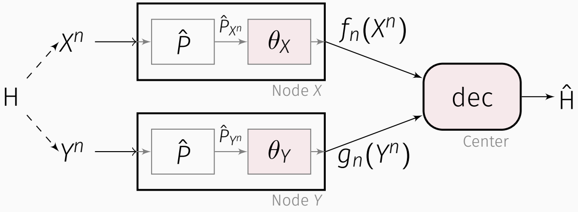

Goal: choose θ_X, θ_Y and dec, such that

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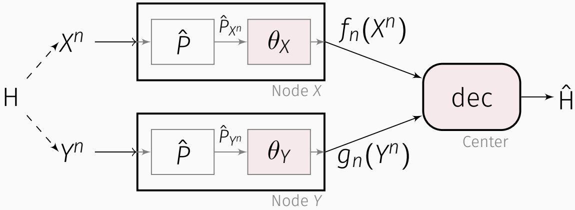

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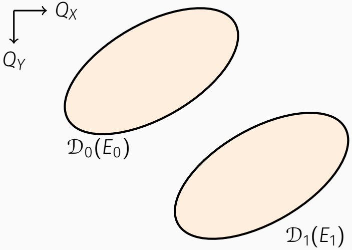
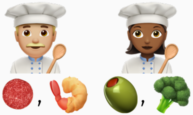
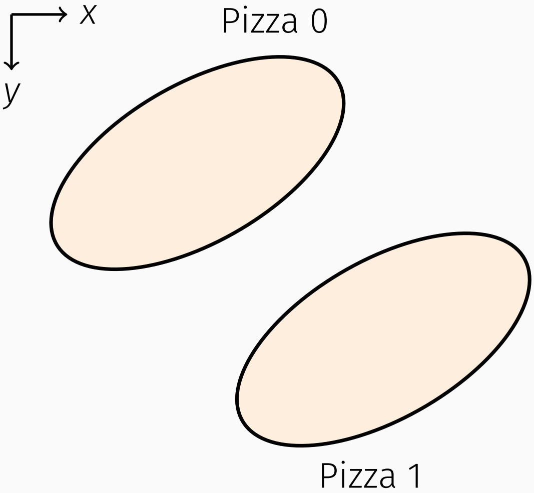
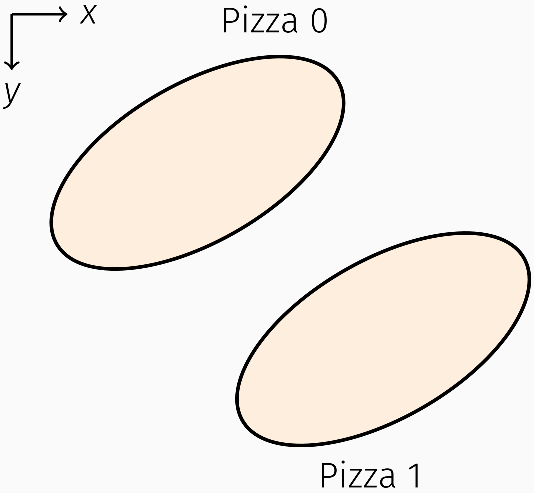


On Distributed Pizza Topping with Finite Ingredients




x
y
Pizza 0
Pizza 1

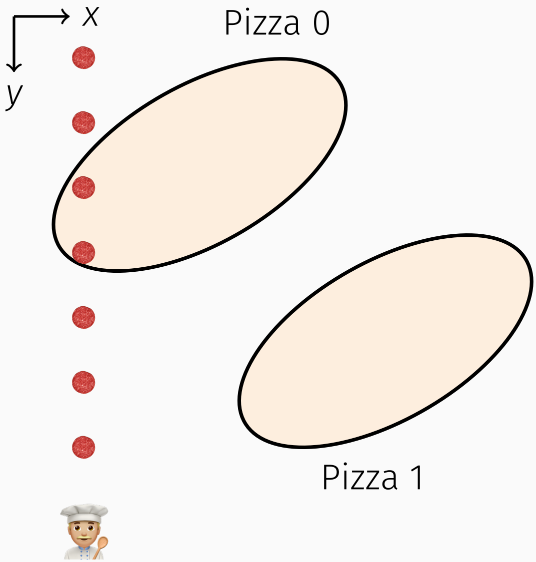
Distributed Pizza Topping



Cook X 👨‍🍳: top based on x

- 

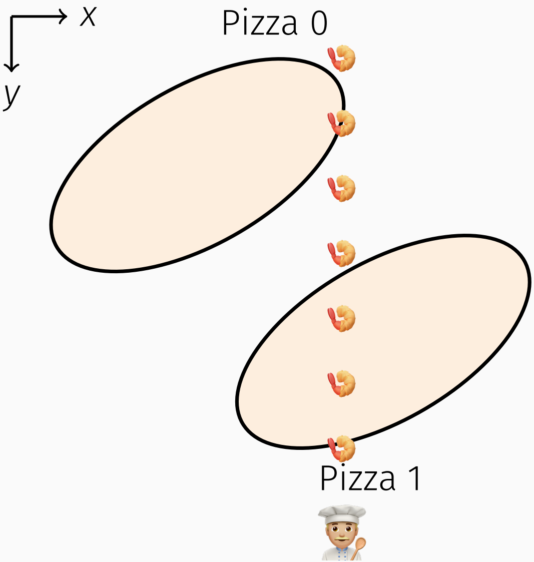
Distributed Pizza Topping



Cook X: top based on x

- ,

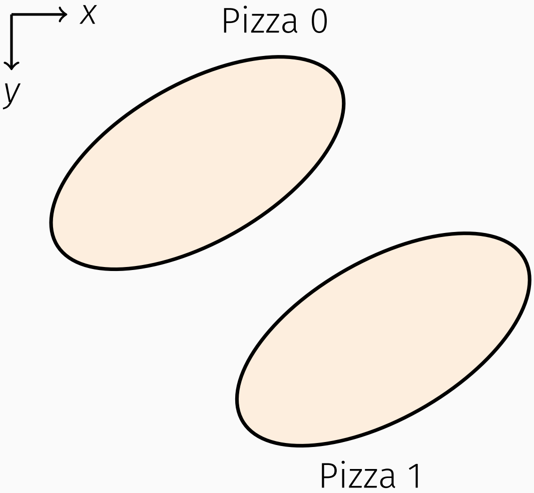
Distributed Pizza Topping



Cook X 🧑‍🍳: top based on x

- 🍕, 🍤

Distributed Pizza Topping

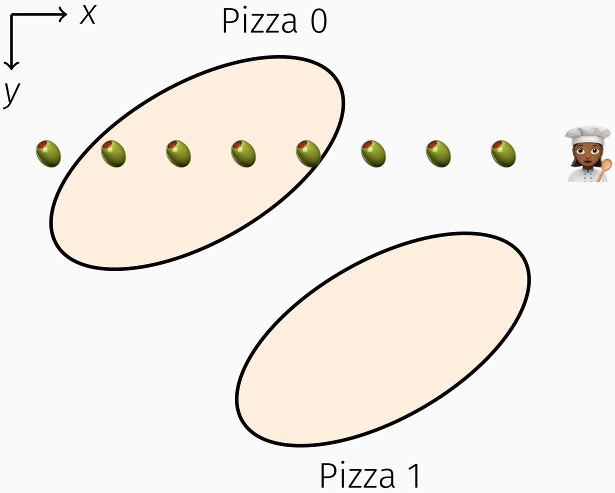


Cook X 👨🏼‍🍳: top based on x

- 🔴, 🦐

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦



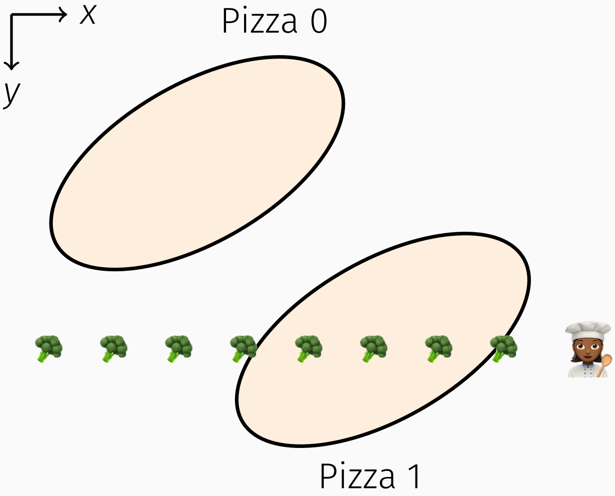
Cook X 👨🏼‍🍳: top based on x

- , 🍤

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦

Distributed Pizza Topping

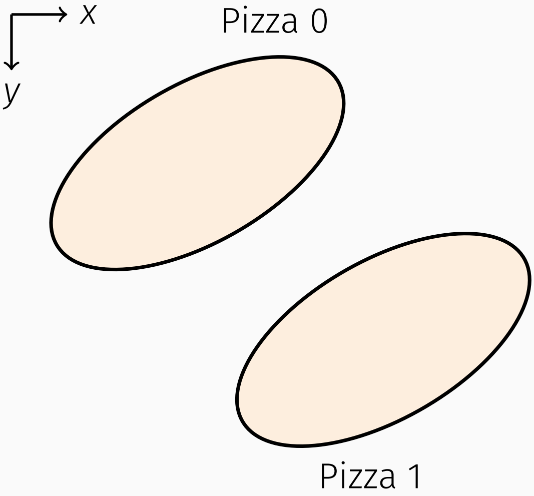


Cook X 🧑‍🍳: top based on x

- 🍕, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦



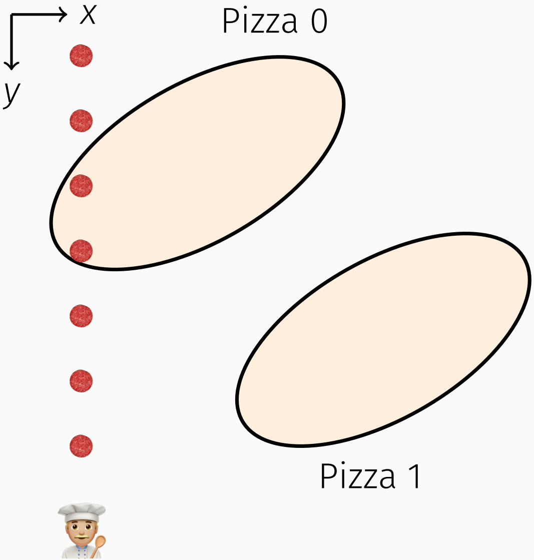
Cook X 👨🏼‍🍳: top based on x

- , 🦐

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas



Cook X 👨🏼‍🍳: top based on x

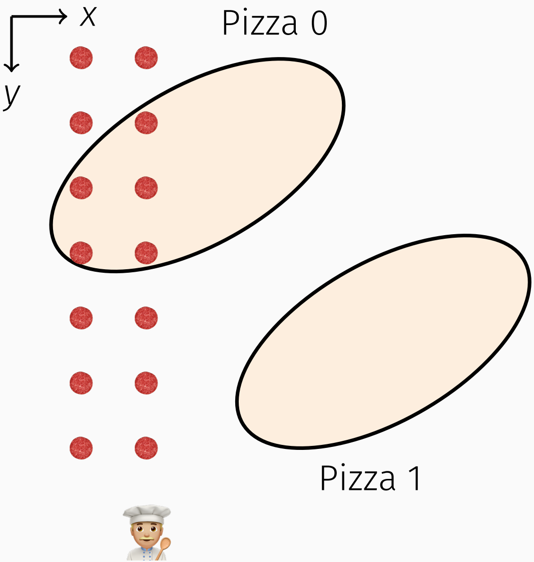
- 🔴, 🍤

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X: top based on x

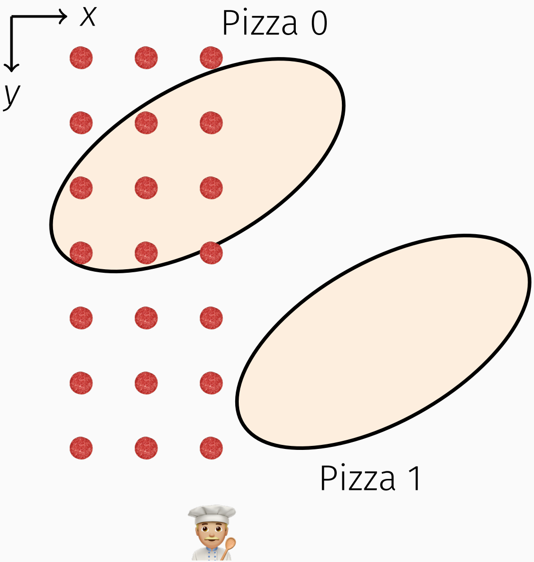
- 🍅, 🍤

Cook Y: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X 👨‍🍳: top based on x

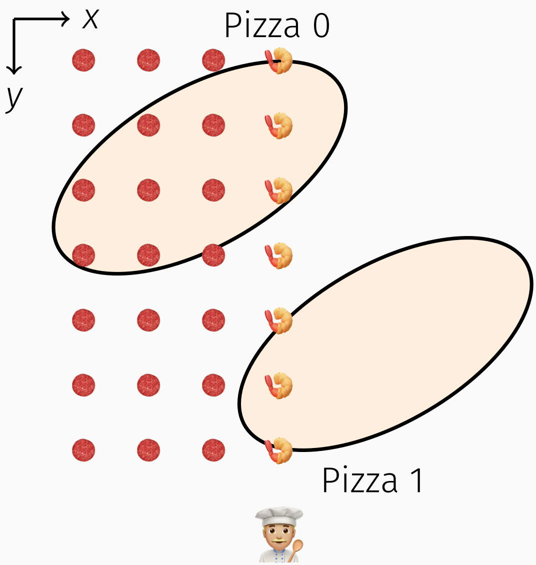
- 🍕, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X 🧑‍🍳: top based on x

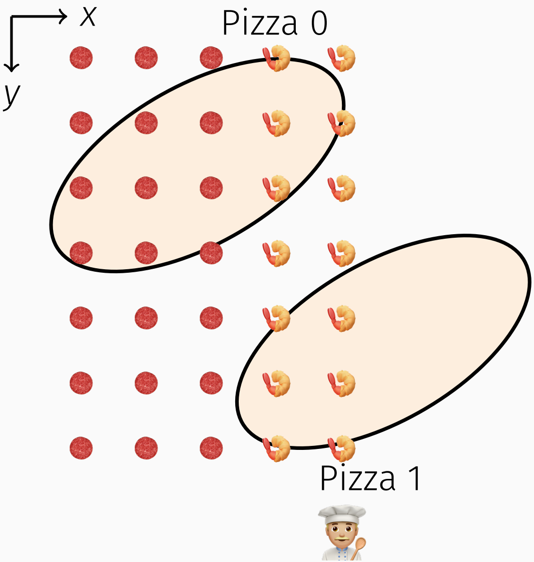
- 🍕, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X: top based on x

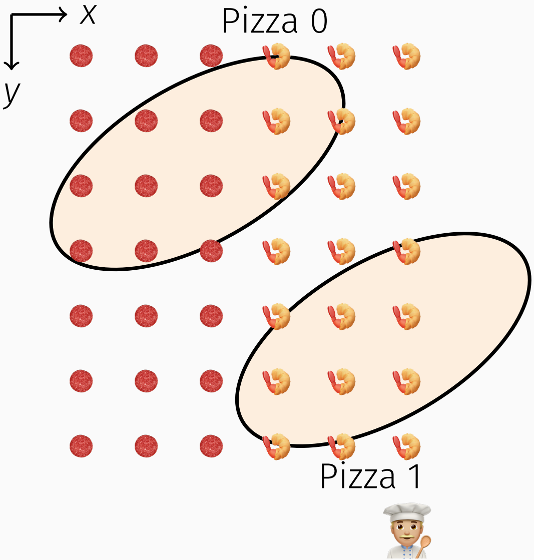
- ,

Cook Y: top based on y

- ,

Goal: distinguishable pizzas

Distributed Pizza Topping



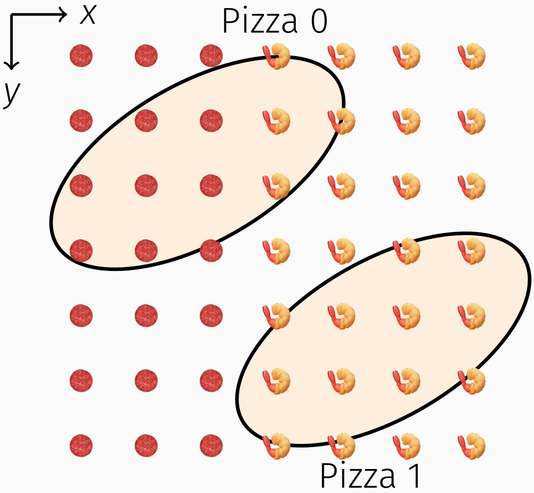
Cook X: top based on x

- ,

Cook Y: top based on y

- ,

Goal: distinguishable pizzas



Cook X 👨‍🍳: top based on x

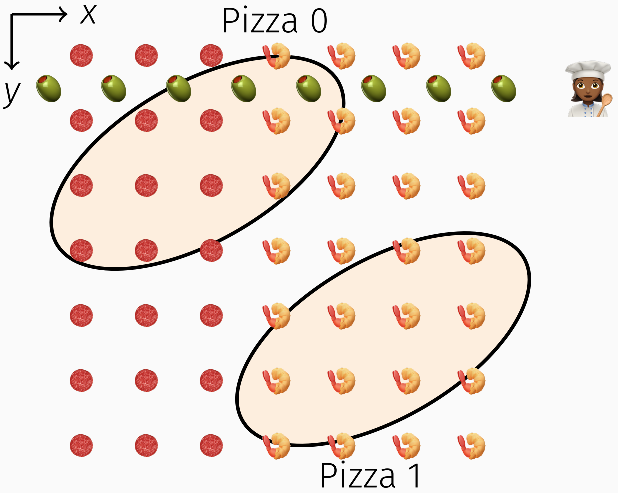
- 🍅, 🦐

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X 🧑‍🍳: top based on *x*

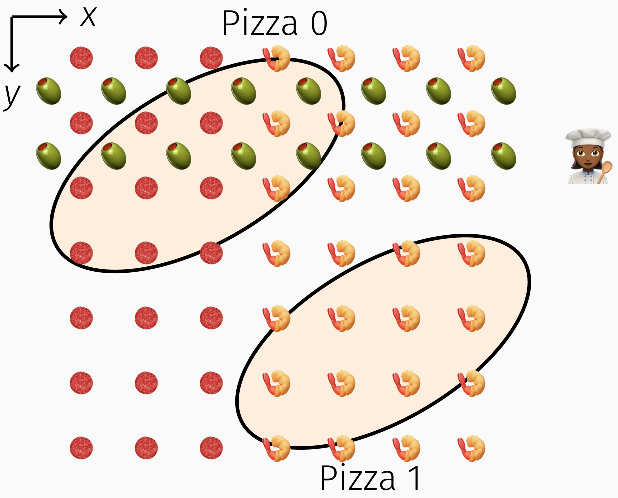
- 🍕, 🍤

Cook Y 👩‍🍳: top based on *y*

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



Cook X: top based on x

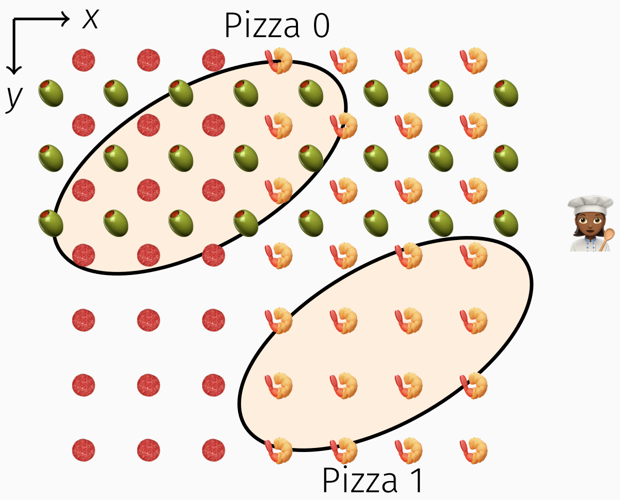
- 🍖, 🍤

Cook Y: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



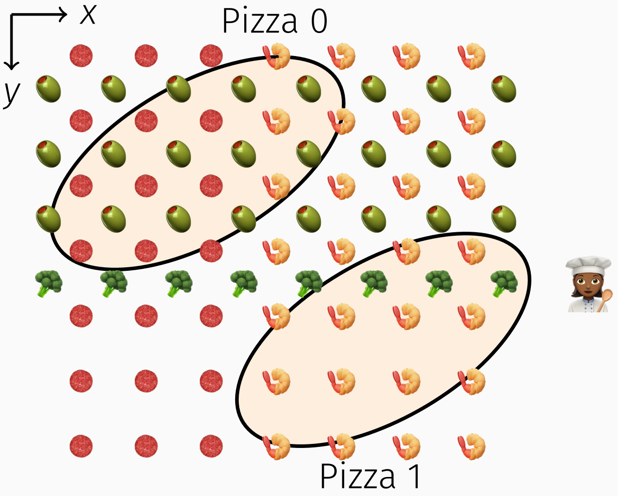
Cook X 🧑‍🍳: top based on x

- 🍕, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas



Cook X 👨‍🍳: top based on x

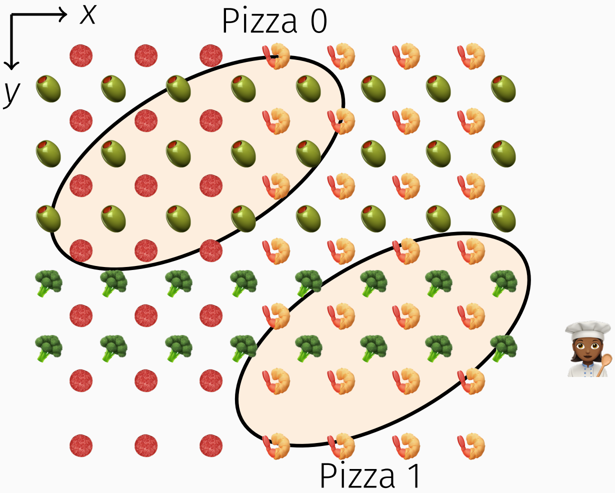
- 🔴, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



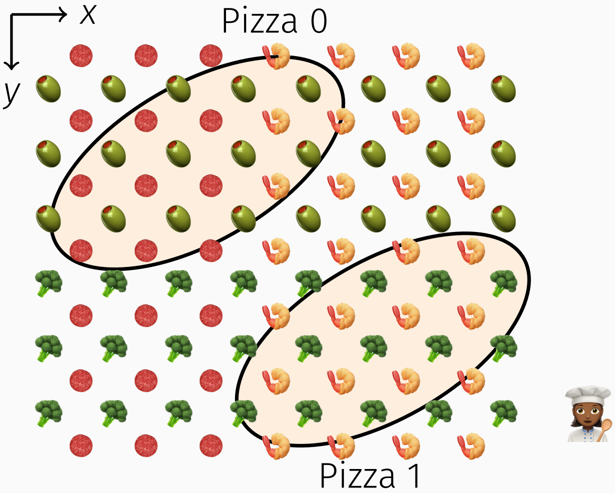
Cook X 👨‍🍳: top based on *x*

- 🔴, 🦐

Cook Y 👩🏾‍🍳: top based on *y*

- 🫒, 🥦

Goal: distinguishable pizzas



Cook X 👨‍🍳: top based on x

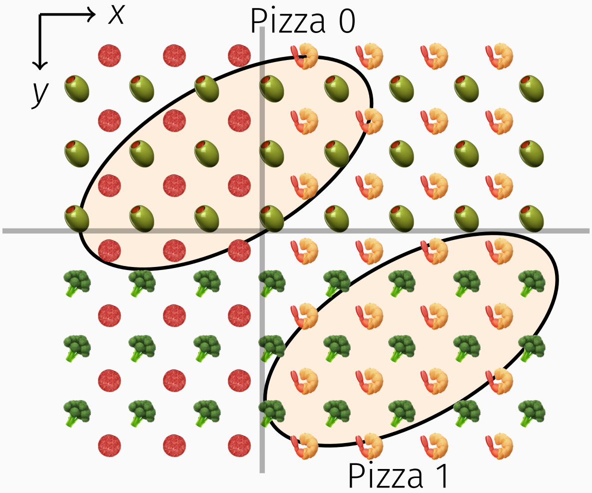
- 🔴, 🍤

Cook Y 👩‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas

Distributed Pizza Topping



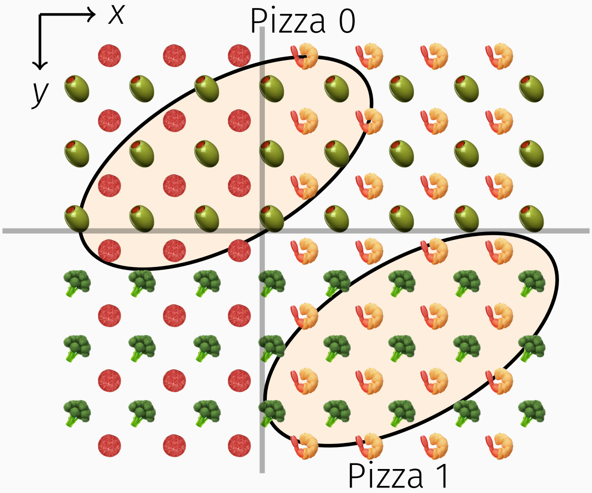
Cook X 👨‍🍳: top based on x

- 🍤

Cook Y 👩🏾‍🍳: top based on y

- 🫒, 🥦

Goal: distinguishable pizzas



Cook X 👨‍🍳: top based on x

- 🔴, 🦐

Cook Y 👩🏾‍🍳: top based on y

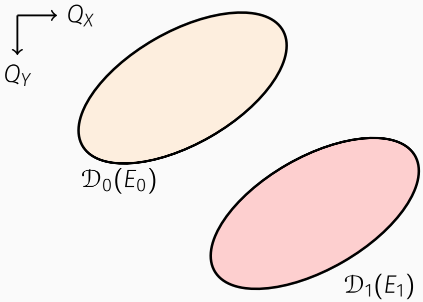
- 🫒, 🥦

Goal: distinguishable pizzas

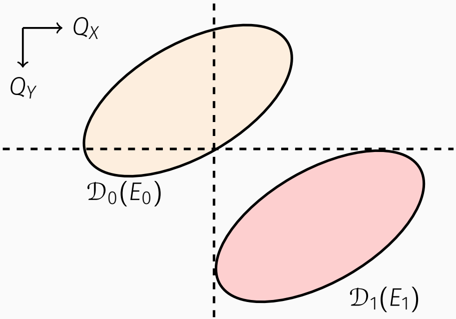
Ingredient Table

	🔴	🦐
🫒	0	0
🥦	0	1

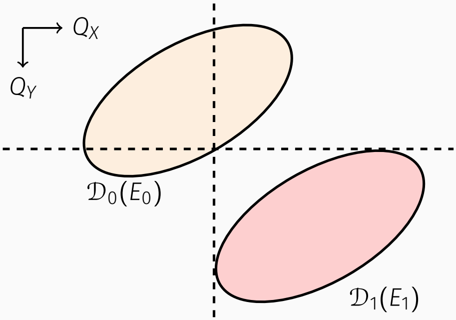
One-bit Compression $M_X = M_Y = 2$



One-bit Compression $M_X = M_Y = 2$

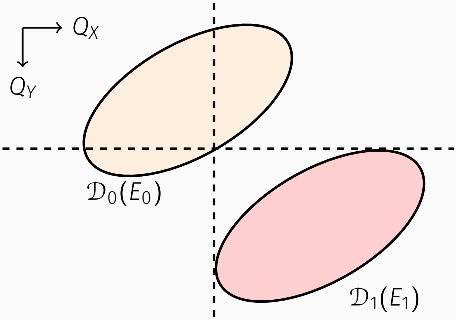


One-bit Compression $M_X = M_Y = 2$



dec	0	1
0	0	0
1	0	1

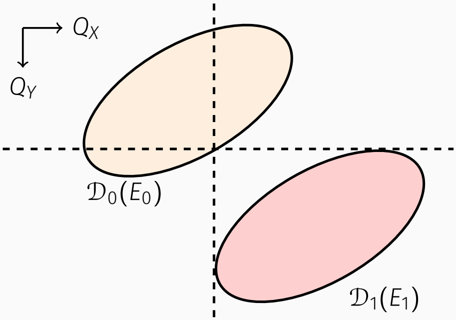
One-bit Compression $M_X = M_Y = 2$



dec	0	1
0	0	0
1	0	1

$\leftrightarrow$

One-bit Compression $M_X = M_Y = 2$

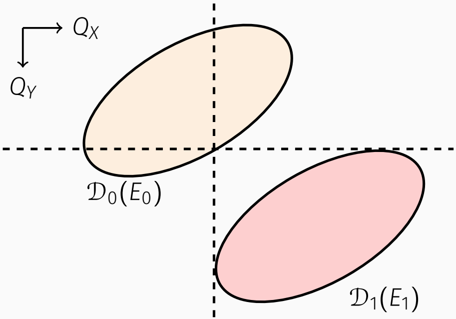


dec	0	1
0	0	0
1	0	1

$\leftrightarrow$

Limiting Case: $M_X = M_Y = \infty$

One-bit Compression $M_X = M_Y = 2$



dec	0	1
0	0	0
1	0	1

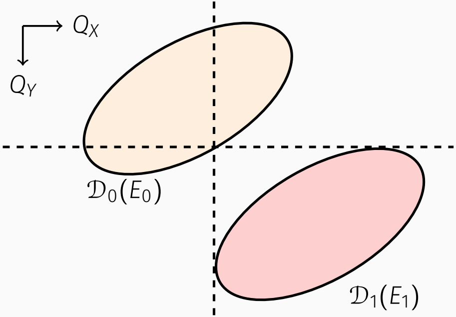
$\leftrightarrow$

Limiting Case: $M_X = M_Y = \infty$

(E_0, E_1) is achievable iff

$$\mathcal{D}_0(E_0) \cap \mathcal{D}_1(E_1) = \varnothing$$

One-bit Compression $M_X = M_Y = 2$



dec	0	1
0	0	0
1	0	1

$\leftrightarrow$

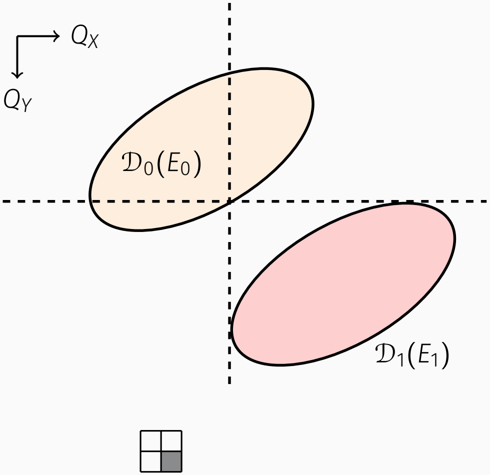
Limiting Case: $M_X = M_Y = \infty$

(E_0, E_1) is achievable iff

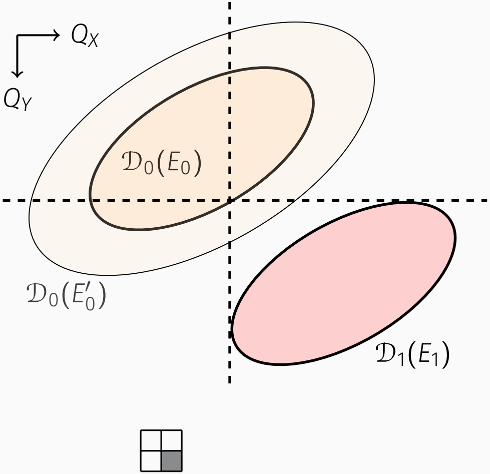
$$\mathcal{D}_0(E_0) \cap \mathcal{D}_1(E_1) = \varnothing$$

zero-rate compression $(0, 0)$

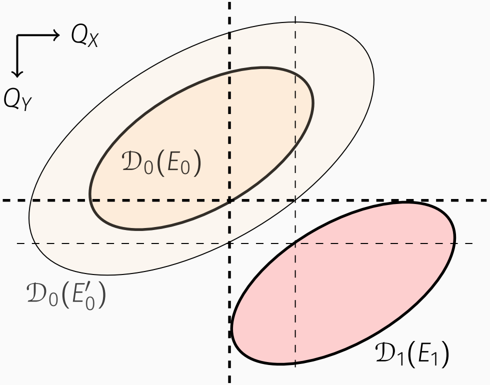
Add more symbols: $(0_2, 0_2) \to (0_3, 0_3)$

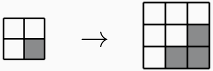


Add more symbols: $(0_2, 0_2) \to (0_3, 0_3)$

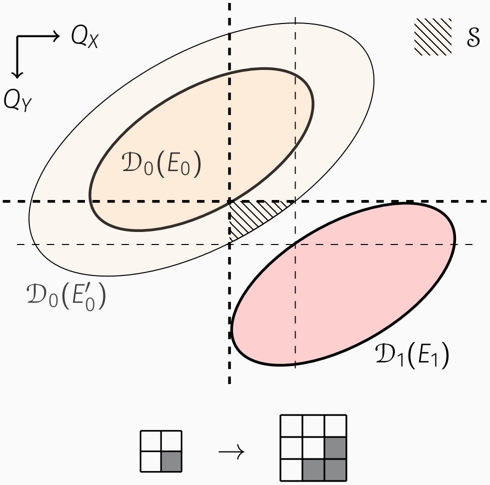


Add more symbols: $(\mathcal{O}_2, \mathcal{O}_2) \to (\mathcal{O}_3, \mathcal{O}_3)$



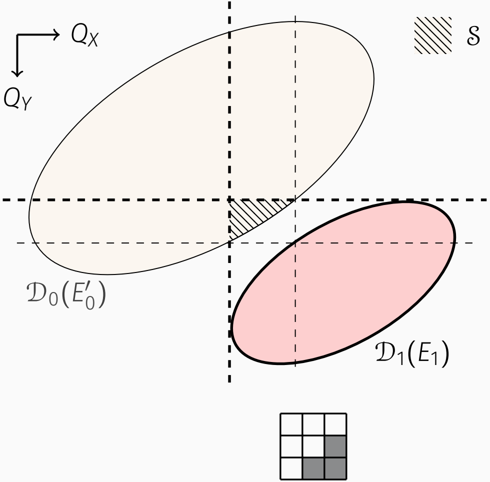


Add more symbols: $(0_2, 0_2) \to (0_3, 0_3)$

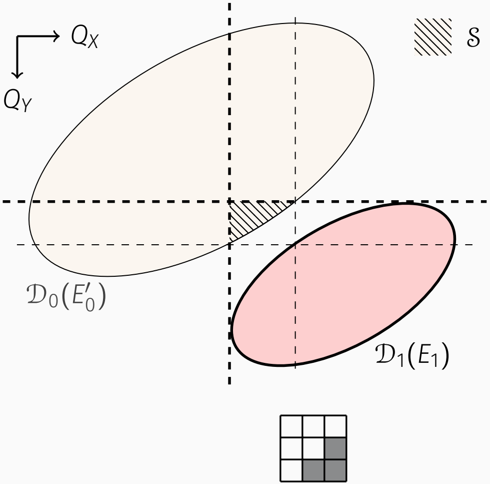


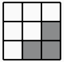
Use added symbols to "describe" $\mathcal{S}$

Add more symbols: $(0_2, 0_2) \to (0_3, 0_3)$



Add more symbols: $(0_2, 0_2) \to (0_3, 0_3)$

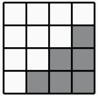


SEPARATE $\mathcal{D}_0(E_0')$ and $\mathcal{D}_1(E_1)$ by 

$$\Updownarrow$$

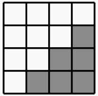
SEPARATE $\mathcal{S}$ and $\mathcal{D}_1(E_1)$ by 

$$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$$



$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

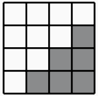
- m_X, m_Y: discrete-valued beliefs



Threshold Decoders

$$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$$

- m_X, m_Y: discrete-valued beliefs
- $m_X + m_Y$: fused belief



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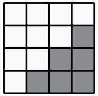
- $M_X \wedge M_Y = \min\{M_X, M_Y\}$: threshold



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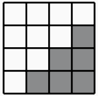
“Reduction”: Recursive Croppings



$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

- m_X, m_Y: discrete-valued beliefs
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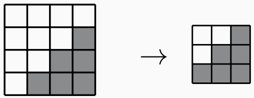
"Reduction": Recursive Croppings



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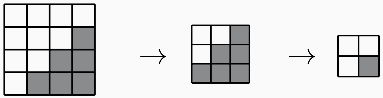
"Reduction": Recursive Croppings



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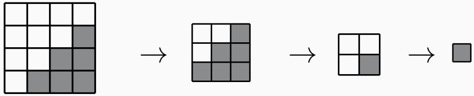
Optimality?

“Reduction”: Recursive Croppings



Threshold Decoders

$$\text{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$$

- m_X, m_Y: discrete-valued beliefs
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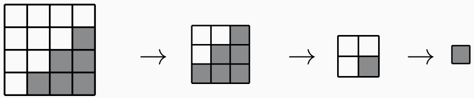
"Reduction": Recursive Croppings



Threshold DECs are Optimal if

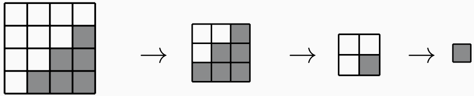
$$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$$

- m_X, m_Y: discrete-valued beliefs
- $m_X + m_Y$: fused belief
- $M_X \wedge M_Y = \min\{M_X, M_Y\}$: threshold

Threshold DECs are Optimal if

$$\min\{M_X, M_Y\} = 2$$

"Reduction": Recursive Croppings



$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

- m_X, m_Y: discrete-valued beliefs
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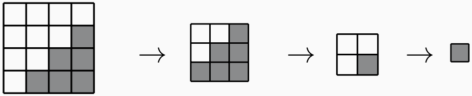
- $M_X \wedge M_Y = \min\{M_X, M_Y\}$: threshold

“Reduction”: Recursive Croppings



Threshold DECs are Optimal if

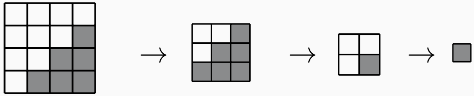
$$\min\{M_X, M_Y\} = 2$$

or $(M_X, M_Y) = (3, 3)$

$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

- m_X, m_Y: discrete-valued beliefs
- $m_X + m_Y$: fused belief
- $M_X \wedge M_Y = \min\{M_X, M_Y\}$: threshold

“Reduction”: Recursive Croppings



Threshold DECs are Optimal if

$\min\{M_X, M_Y\} = 2$

or $(M_X, M_Y) = (3, 3)$

or $P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}$ for some $i \in \{0, 1\}$

$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

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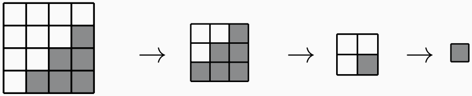
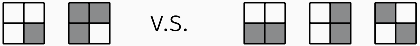
“Reduction”: Recursive Croppings



Example: $M_X = M_Y = 2$



$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

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Threshold DECs are Optimal if

$\min\{M_X, M_Y\} = 2$

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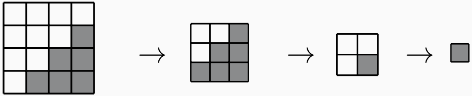
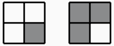
or $P^{(i)}_{XY} = P^{(i)}_X P^{(i)}_Y$ for some $i \in \{0, 1\}$

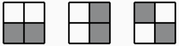
"Reduction": Recursive Croppings



Example: $M_X = M_Y = 2$



v.s.



What if $M_X = M_Y = 4$?

Threshold Decoders

$\mathrm{dec}(m_X, m_Y) = \mathbb{1}\{m_X + m_Y \geq M_X \wedge M_Y\}$

- m_X, m_Y: discrete-valued beliefs
- $m_X + m_Y$: fused belief
- $M_X \wedge M_Y = \min\{M_X, M_Y\}$: threshold

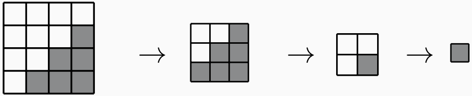
Threshold DECs are Optimal if

$$\min\{M_X, M_Y\} = 2$$

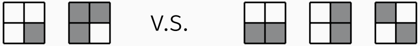
or $(M_X, M_Y) = (3, 3)$

or $P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}$ for some $i \in \{0, 1\}$

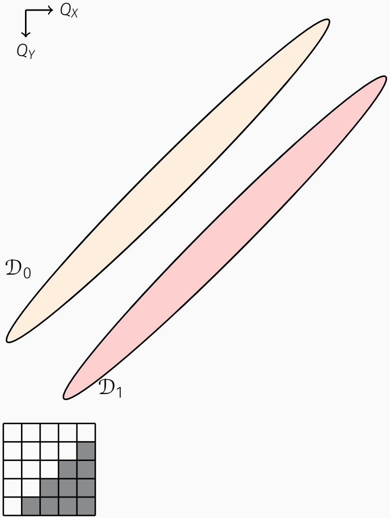
"Reduction": Recursive Croppings



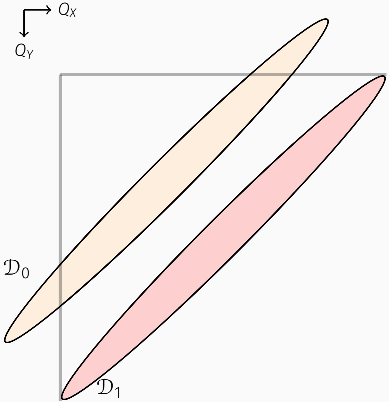
Example: $M_X = M_Y = 2$

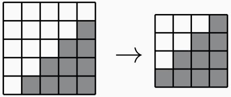


What if $M_X = M_Y = 4$?

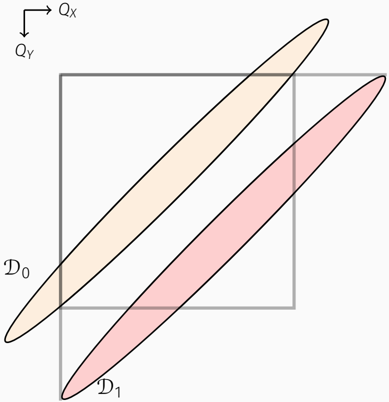

Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$

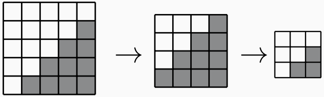
Baguettes 🥖🥖



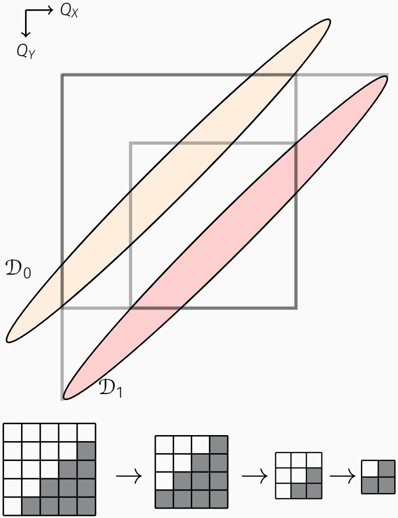


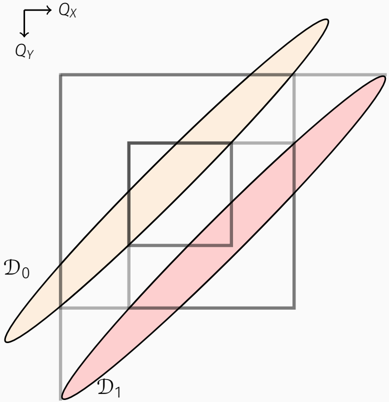
Baguettes 🥖🥖

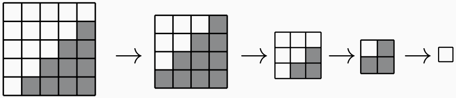




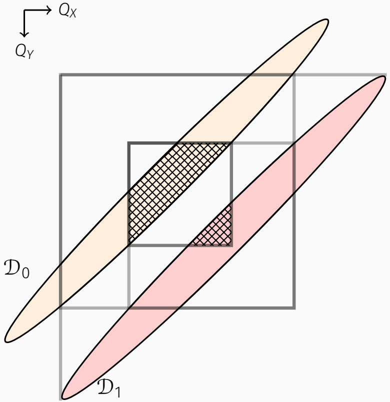
Baguettes 🥖🥖

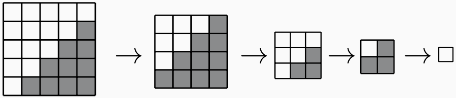



Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$

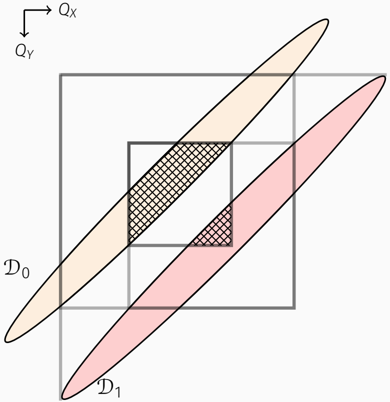


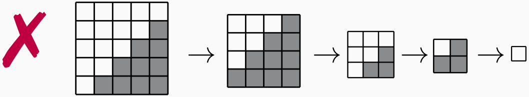
Baguettes 🥖🥖





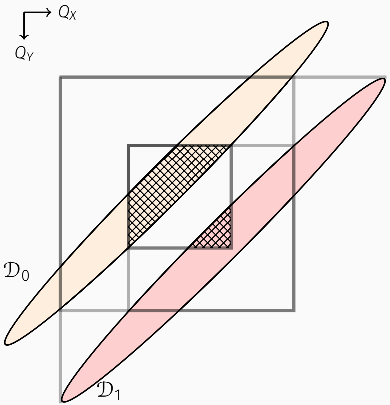
BAGUETTES 🥖🥖

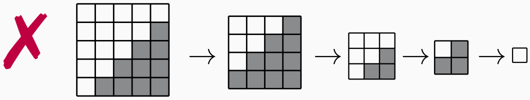




Threshold DEC requires $\geq$ 6 Symbols

Baguettes 🥖🥖

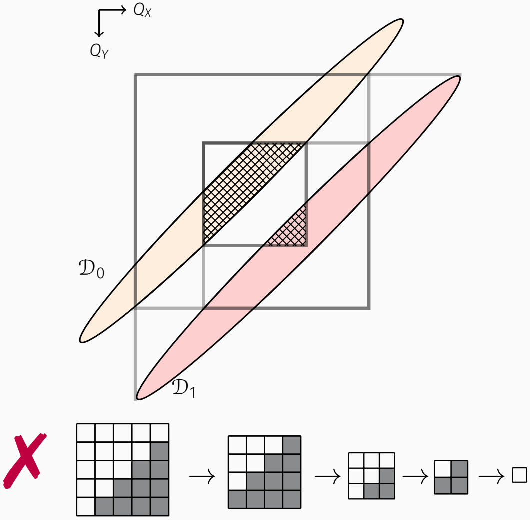


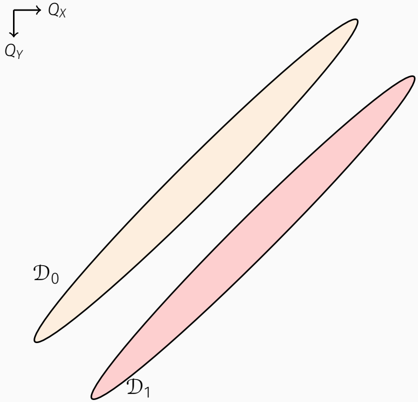


Threshold DEC requires $\geq$ 6 Symbols

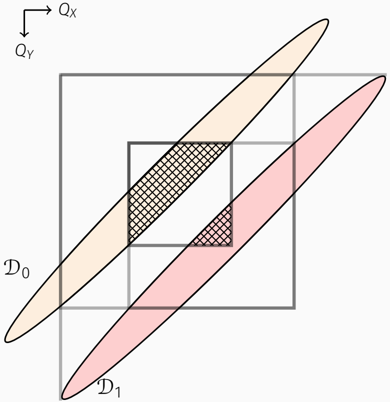
Can we do better?

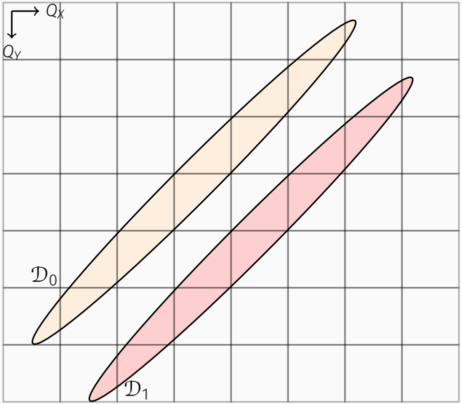
Baguettes 🥖🥖

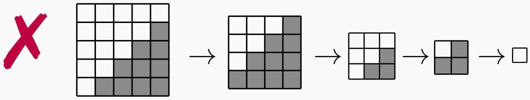




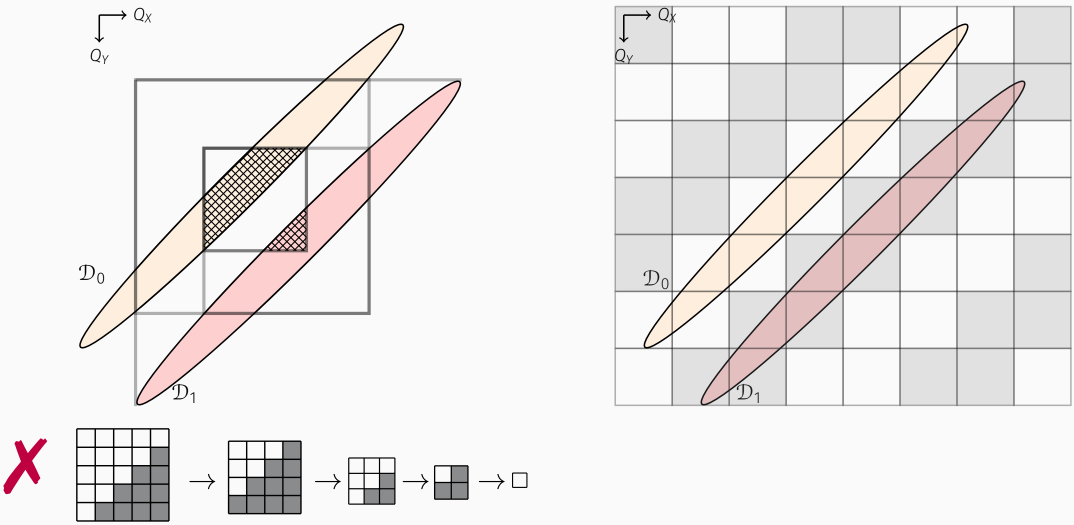
Baguettes 🥖🥖



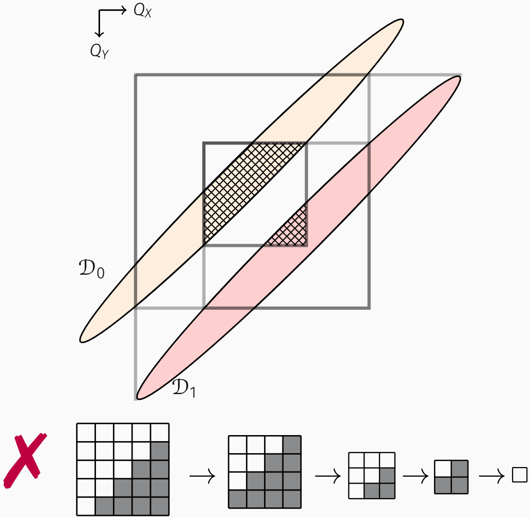




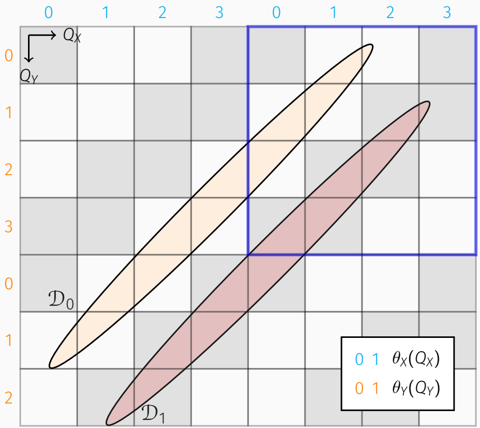
Baguettes 🥖🥖



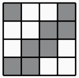
Baguettes 🥖🥖



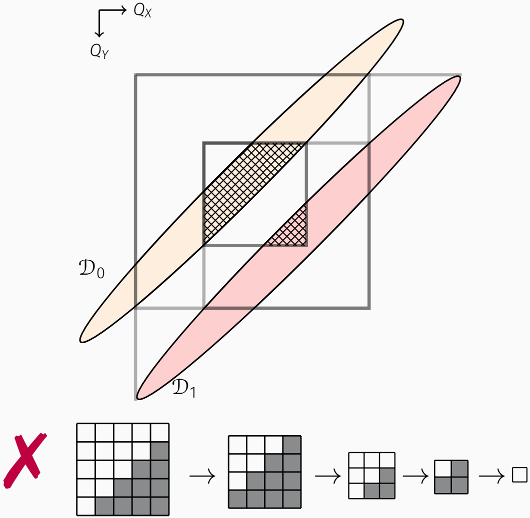
Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$



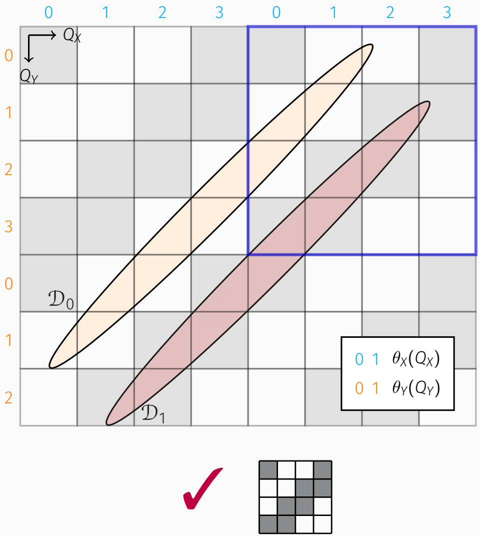
0 1 2 3 0 1 2 3
0 1 2 3 0 1 2
Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$
0 1 $\theta_X(Q_X)$
0 1 $\theta_Y(Q_Y)$



Baguettes 🥖🥖



Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$
✗
→ → → →



0 1 2 3 0 1 2 3
0 1 2 3 0 1 2
Q_X
Q_Y
$\mathcal{D}_0$
$\mathcal{D}_1$
0 1 $\theta_X(Q_X)$
0 1 $\theta_Y(Q_Y)$
✓

Numerical Example I

$$\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$$

$$P_X^{(0)} = P_Y^{(0)} = (1/8, 1/8, 3/4)$$

$$P_X^{(1)} = P_Y^{(1)} = (3/8, 3/8, 1/4)$$

$$P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}, i \in \{0, 1\}$$

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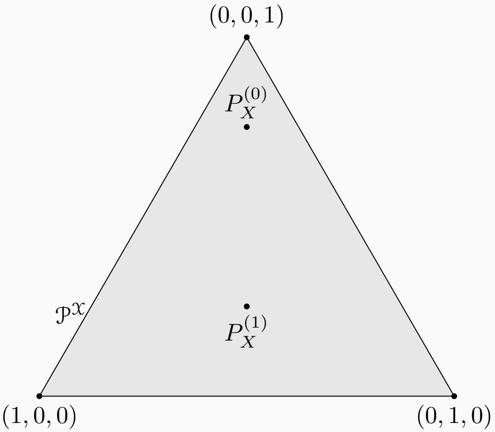

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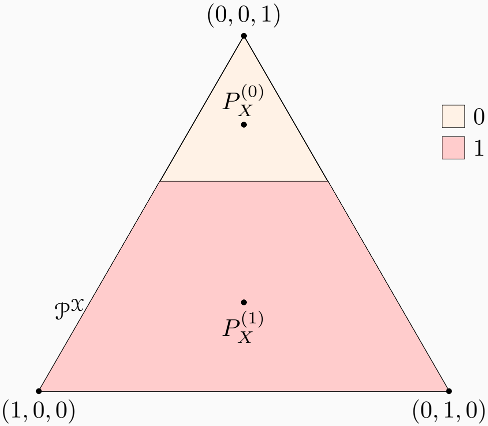
$$P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}, i \in \{0, 1\}$$

Local Decision at X: Neyman-Pearson



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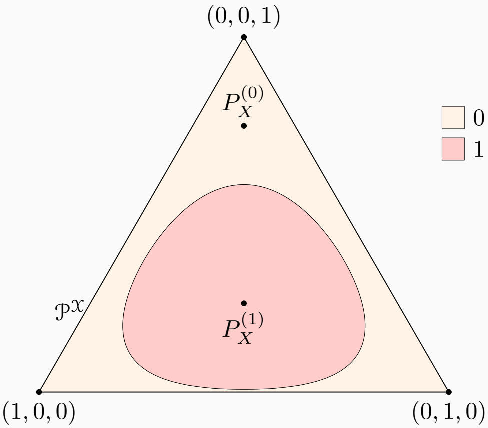
$$P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}, i \in \{0, 1\}$$

Local Decision at X: Hoeffding (Generalized Likelihood Ratio Test)



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$$P_{XY}^{(i)} = P_X^{(i)} P_Y^{(i)}, i \in \{0, 1\}$$

- Comm. Constraints: $(0_7, 0_7)$

Numerical Example I

$$\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$$
$$P_X^{(0)} = P_Y^{(0)} = (1/8, 1/8, 3/4)$$
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- Comm. Constraints: $(0_7, 0_7)$
- dec: $(m_X, m_Y) \mapsto \mathbb{1}\{m_X + m_Y \geq 7\}$

Numerical Example I

$$\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$$
$$P_X^{(0)} = P_Y^{(0)} = (1/8, 1/8, 3/4)$$
$$P_X^{(1)} = P_Y^{(1)} = (3/8, 3/8, 1/4)$$
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- Comm. Constraints: $(0_7, 0_7)$
- dec: $(m_X, m_Y) \mapsto \mathbb{1}\{m_X + m_Y \geq 7\}$
- $(E_0, E_1) = (0.3, 0.25)$

Numerical Example I

$$\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$$
$$P_X^{(0)} = P_Y^{(0)} = (1/8, 1/8, 3/4)$$
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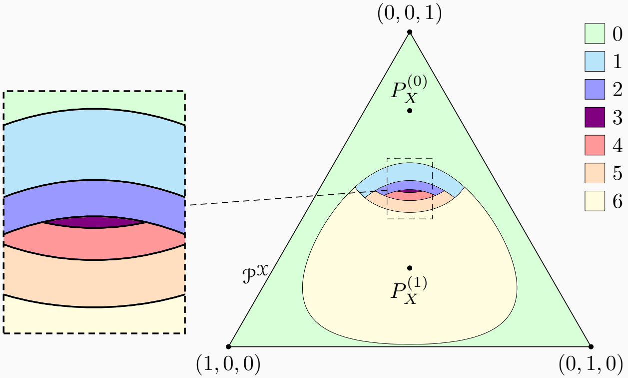
DHT: Type-encoding Function

$$\theta_X \colon \mathcal{P}^{\mathcal{X}} \to \{0, 1, \ldots, 6\}$$



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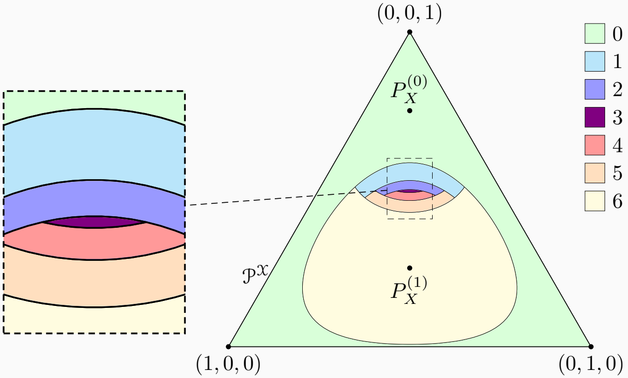
DHT: Type-encoding Function

$$\theta_X : \mathcal{P}^{\mathcal{X}} \to \{0, 1, \ldots, 6\}$$



Quantization of LLR is not optimal!

Numerical Example II: Error Exponents

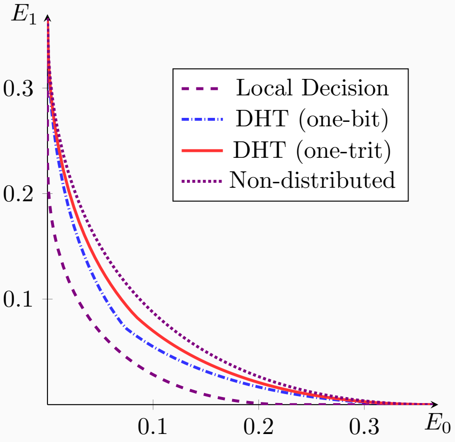
$\mathcal{X} = \mathcal{Y} = \{0, 1\}$

$$P_{XY}^{(0)} = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/2 \end{bmatrix}, P_{XY}^{(1)} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$$

Numerical Example II: Error Exponents

$$\mathcal{X} = \mathcal{Y} = \{0, 1\}$$

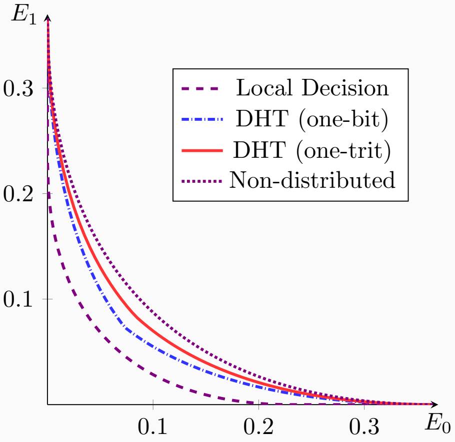
$$P_{XY}^{(0)} = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/2 \end{bmatrix}, P_{XY}^{(1)} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$$



$\mathcal{X} = \mathcal{Y} = \{0, 1\}$

$$P_{XY}^{(0)} = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/2 \end{bmatrix}, P_{XY}^{(1)} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$$

- Local Decision: $(0_1, \log 2)$ or $(\log 2, 0_1)$



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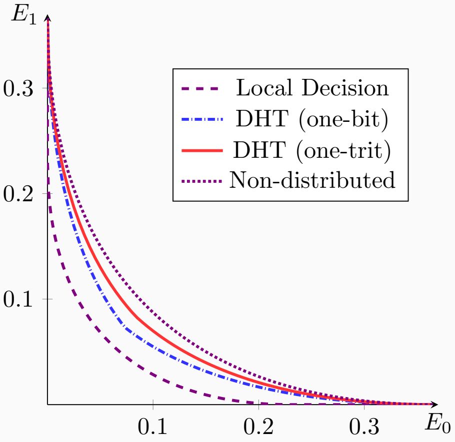
- Local Decision: $(0_1, \log 2)$ or $(\log 2, 0_1)$
- One-bit: $(0_2, 0_2)$



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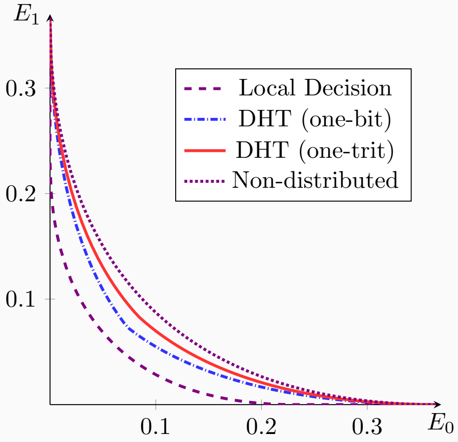
- Local Decision: $(0_1, \log 2)$ or $(\log 2, 0_1)$
- One-bit: $(0_2, 0_2)$
- One-trit: $(0_3, 0_3)$



Numerical Example II: Error Exponents

$$\mathcal{X} = \mathcal{Y} = \{0, 1\}$$

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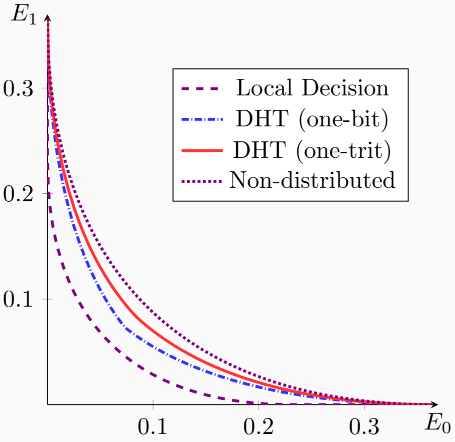
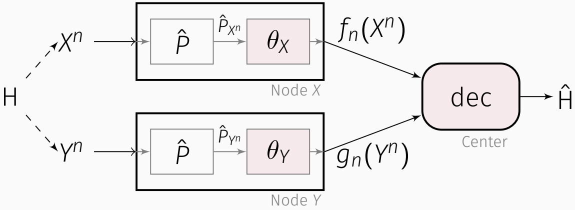
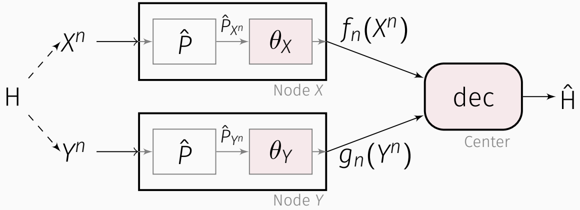
- Local Decision: $(0_1, \log 2)$ or $(\log 2, 0_1)$
- One-bit: $(0_2, 0_2)$
- One-trit: $(0_3, 0_3)$
- Non-distributed: $(\log 2, \log 2)$

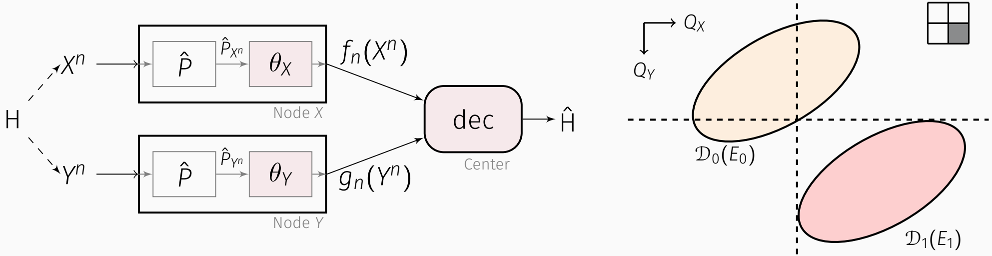


SUMMARY: DHT WITH CONSTANT COMMUNICATION BITS



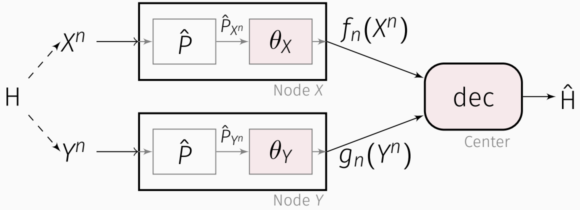


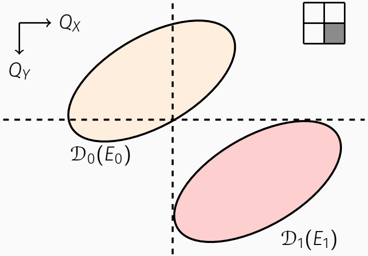
- Optimality of Encoding Types: Sequence Space $\rightarrow$ Distribution Space



- Optimality of Encoding Types: Sequence Space $\rightarrow$ Distribution Space
- Geometric Interpretation: Separating Divergence Balls

SUMMARY: DHT WITH CONSTANT COMMUNICATION BITS





- Optimality of Encoding Types: Sequence Space $\to$ Distribution Space
- Geometric Interpretation: Separating Divergence Balls
- Threshold Decoders: Error Exponents & Optimality