Sequential Dependence Decomposition & Feature Learning

Xiangxiang Xu
Joint work with Prof. Lizhong Zheng

Allerton Conference 2023

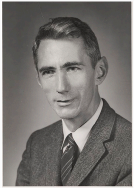
Sequences

- 

- Videos
- `exampe of dependnce in plai tetx`

Guessing Next Letter

1. E $\Longrightarrow$ solve E ?



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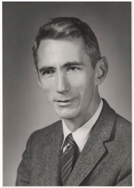

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  - "E" followed by N: 0.5, P: 0.25, 'SPC': 0.25

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Shannon's Experiments [1948]

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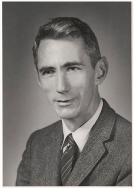

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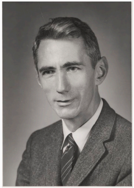

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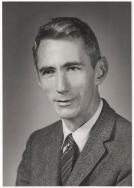

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- predict based on the last letter and digram frequency
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  - word-level construction
  - trigram, *n*-gram (higher order Markov chains)

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Guessing Next Letter

Second-order approximation (digram structure as in English).

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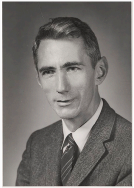

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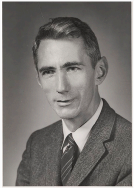

Operations

- frequency tables

264 SECRET AND URGENT

TABLE XII

English Trigrams

Figures represent approximate frequencies for 20,000 words, but are more important in their relation to one another.

A		agn	2	ame	53	aro	7
Aam	1	ago	6	amh	1	arp	1
		agr	8	ami	6	arr	19
Aba	6	agu	2	aml	1	ars	46
abd	1			amn	1	art	48
abe	1	Ahe	2	amo	8	arv	3
abi	3	aho	1	amp	10	ary	34
abl	39	ahu	1	ams	6		
abo	28	ahy	1	amy	1		
abr	1					Asa	1
abs	1	Aid	24	Ana	6	asc	3
		aig	3	anc	39	ase	20

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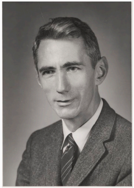

Operations

- frequency tables
- *opens a book...*
  select a letter at random

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- n-gram table on $\mathcal{X} \implies |\mathcal{X}|^n$ entries
- “book”: may not contain all combinations

Markov Approximation

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Deep Learning Solution

- entries are structured

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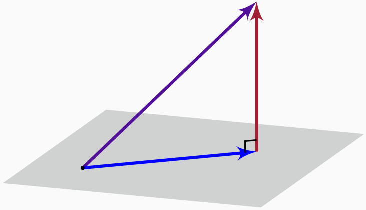
Hidden Parts?

- black-box features
- dependence structure, e.g., the "order"

Feature Geometry



$(X, Y) \sim P_{X,Y}$ variables of interest, e.g., input and output pair

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- Features of X: $\mathcal{F}_{\mathcal{X}} \triangleq \{\mathcal{X} \to \mathbb{R}\}$

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- $f \otimes g \triangleq \sum_{i=1}^{k} f_i \otimes g_i$ for k-dim features

$f = (f_1, \ldots, f_k)$
$g = (g_1, \ldots, g_k)$

Canonical Dependence Kernel (CDK)

$$\mathfrak{i}_{X;Y} = \frac{P_{X,Y}}{P_X P_Y} - 1$$

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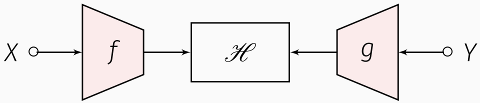

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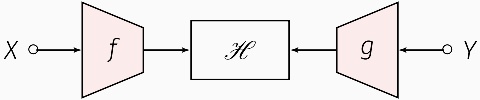

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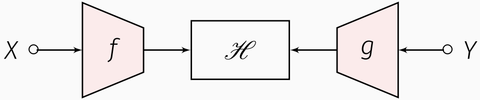

- maximize $\mathscr{H}(f,g) \quad \Longrightarrow \quad f \otimes g = \mathfrak{i}_{X;Y}$
  - compute $\|\mathfrak{i}_{X;Y}\|$ from features
  - prediction/estimation
    $P_{X|Y}(x|y) = P_X(x) \cdot [1 + f(x) \cdot g(y)]$

One More Variable

Learn X based on

- Y: $\mathbf{i}_{X;Y}$

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- (Y, Z): $\mathfrak{i}_{X;Y,Z}$

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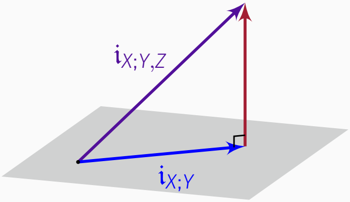

Markov Plane

- $\mathfrak{i}$: $X - Y - Z$

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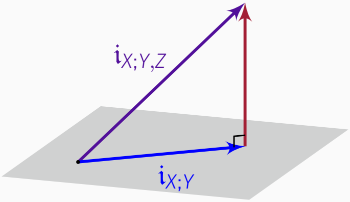
$\mathfrak{i}_{X;Y}$ Markov Component



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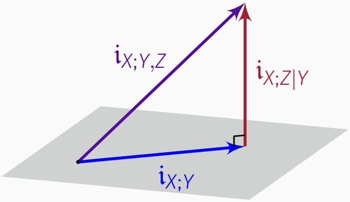
$$\mathfrak{i}_{X;Z|Y} \triangleq \mathfrak{i}_{X;Y,Z} - \mathfrak{i}_{X;Y}$$

$\mathfrak{i}_{X;Y}$ Markov Component

$\mathfrak{i}_{X;Z|Y}$ Conditional Dependence



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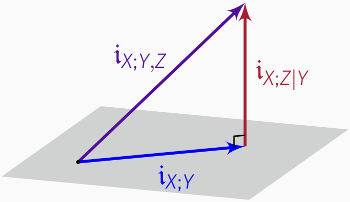

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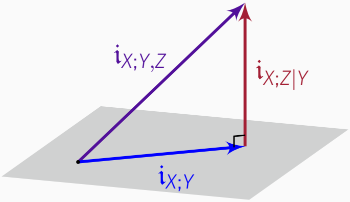

Markov Plane

- $\mathfrak{i}$: $X - Y - Z$

$$\|\mathfrak{i}_{X;Y,Z}\|^2 = \|\mathfrak{i}_{X;Y}\|^2 + \|\mathfrak{i}_{X;Z|Y}\|^2$$

Learning Conditional Dependence

$\mathfrak{i}_{X;Z|Y} = \mathfrak{i}_{X;Y,Z} - \mathfrak{i}_{X;Y}$: dependence Y cannot capture

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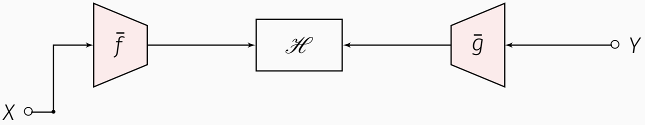

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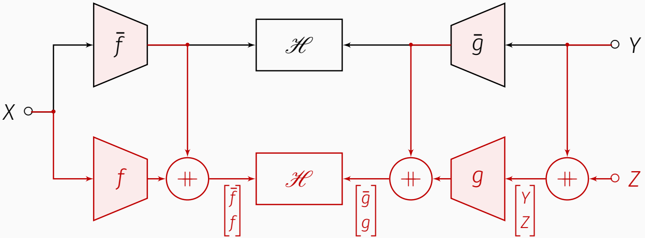

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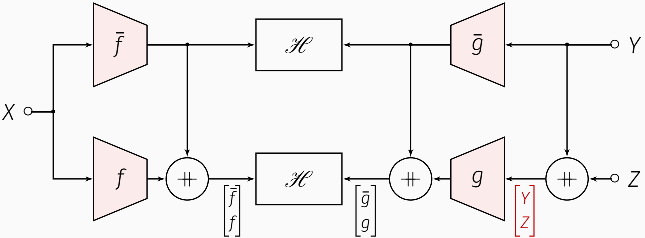

- nesting: separate conditional dependence from the joint

Learning Conditional Dependence

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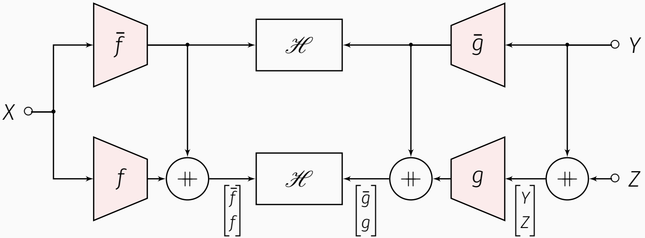

- nesting: separate conditional dependence from the joint
- training: maximize the sum of two H-scores
  - optimal solution: $\bar{f} \otimes \bar{g} = \mathfrak{i}_{X;Y}$, $f \otimes g = \mathfrak{i}_{X;Z|Y}$
  - measure the strength of conditional dependence

Sequential Dependence Decomposition

$$\cdots \quad X_{-3} \quad X_{-2} \quad X_{-1} \quad X_0 \quad \cdots$$

$X_{-1} \quad X_0 \quad \cdots$

"Looking Back"

Learn X_0 based on

- previous state: $\mathfrak{i}_{X_0;X_{-1}}$

$X_{-2} \quad X_{-1} \quad X_0 \quad \cdots$

"Looking Back"

Learn X_0 based on

- previous state: $\mathfrak{i}_{X_0;X_{-1}}$
- past 2 states: $\mathfrak{i}_{X_0;(X_{-1},X_{-2})}$

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Learn X_0 based on

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$\vdots$

- past n states: $\mathfrak{i}_{X_0;(X_{-1},\cdots,X_{-n})}$

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- previous state: $\mathfrak{i}_{X_0;X_{-1}}$
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- Gain from ℓ-th layer $\quad \mathfrak{i}_\ell \triangleq \mathfrak{i}_{X_0;X_{-\ell}|(X_{-1},\cdots,X_{-l+1})}$
  - conditional dependence at lag ℓ

$$\cdots \quad X_{-3} \quad X_{-2} \quad X_{-1} \quad X_0 \quad \cdots$$

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Learn X_0 based on

- previous state: $\mathfrak{i}_{X_0;X_{-1}}$
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  ⋮

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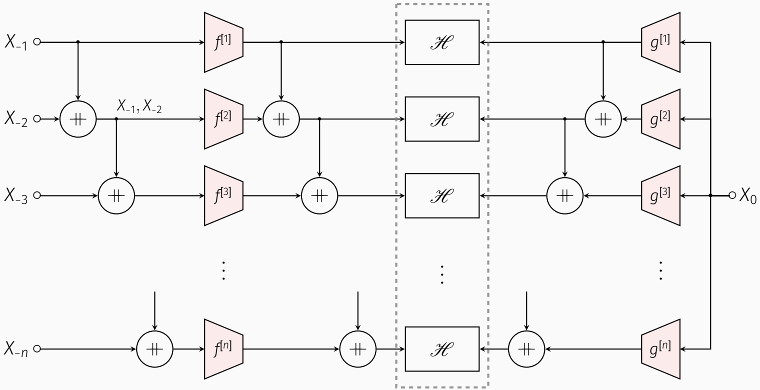
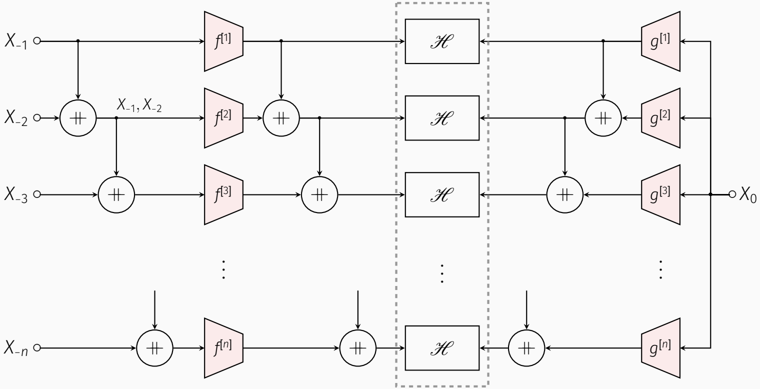
- Gain from ℓ-th layer $\quad \mathfrak{i}_\ell \triangleq \mathfrak{i}_{X_0;X_{-\ell}|(X_{-1},\cdots,X_{-l+1})}$
  - conditional dependence at lag ℓ
- Orthogonal decomposition
  - Dependence between X_0 and past n states $= \sum_{\ell=1}^{n} \mathfrak{i}_\ell$

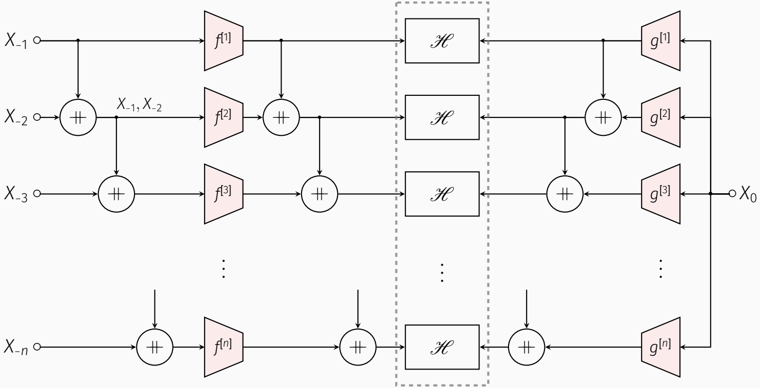
Learning the Decomposition



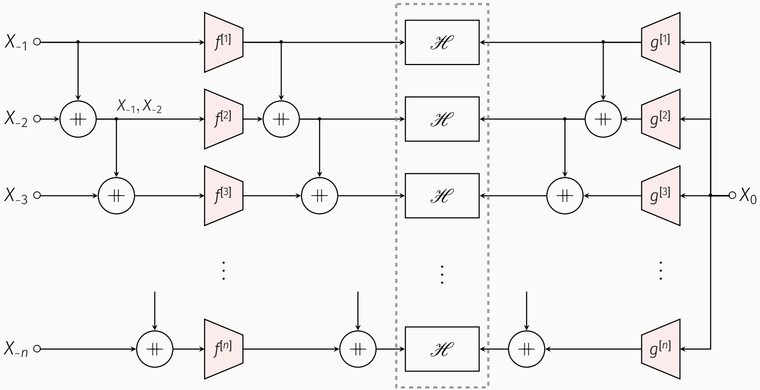


- ℓ-th branch learns $\mathfrak{i}_\ell$: conditional dependence at lag ℓ

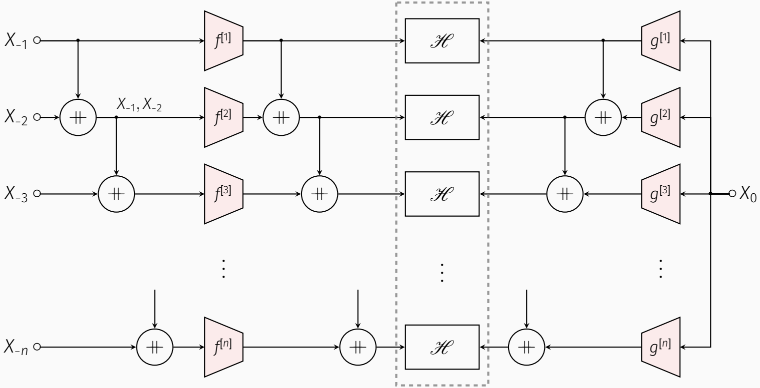
Learning the Decomposition



- ℓ-th branch learns $\mathfrak{i}_\ell$: conditional dependence at lag ℓ
  - top ℓ branches: dependence between X_0 and past ℓ states



- ℓ-th branch learns $\mathbf{i}_\ell$: conditional dependence at lag ℓ
  - top ℓ branches: dependence between X_0 and past ℓ states
- dependence "spectrum" over lags: $\{\|\mathbf{i}_\ell\|^2, \ell \geq 1\}$



- ℓ-th branch learns $\mathfrak{i}_\ell$: conditional dependence at lag ℓ
  - top ℓ branches: dependence between X_0 and past ℓ states
- dependence "spectrum" over lags: $\{\|\mathfrak{i}_\ell\|^2, \ell \geq 1\}$
  - Markov Chain of Order $M \implies$ cutoff at $\ell = M$

Sequence Observations

- Dependence on the History?

Demo

Sequence Observations

- Dependence on the History?
- First/Second/Third-Order?

Demo

Sequence Observations

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- ⋆ Plot Dependence Spectrum $\|\mathbf{i}_\ell\|^2$, $\ell \geq 1$

Sequence Observations

- ▶ Dependence on the History?
- ▶ First/Second/Third-Order?
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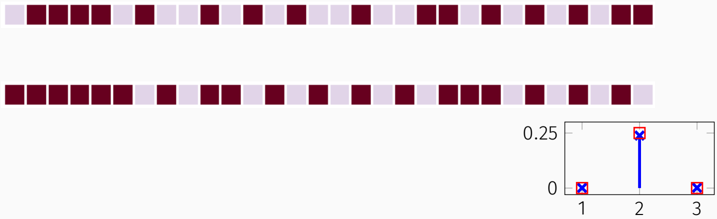

Sequence Observations

- ▶ Dependence on the History?
- ▶ First/Second/Third-Order?
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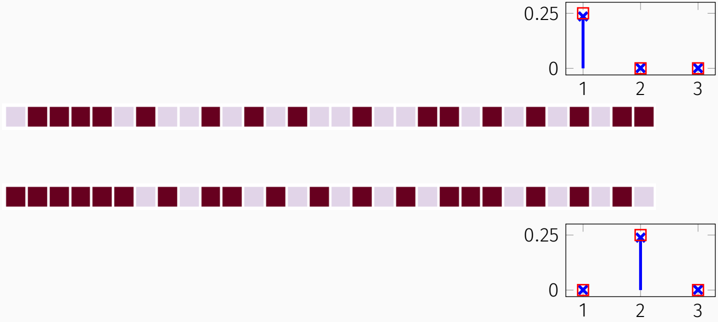

Summary

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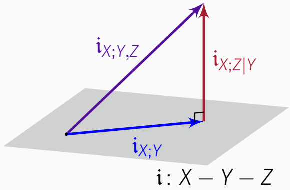

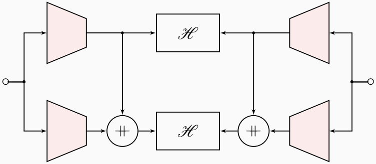

- Feature Geometry
  - Feature Learning $\leftrightarrow$ Geometric Operations
  - Nesting Technique
- Case Study: Learning Random Processes
  - Decompose Sequential Dependence

Learn More

- arXiv: 2309.10140